Name: ______________________________ **Section Number:**______________

110.211 HONORS MULTIVARIABLE CALCULUS
SPRING 2012
MIDTERM EXAMINATION Solutions
April 25, 2012

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		15
2		20
3		15
4		15
5		20
6		15
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:**______________

Question 1. [15 points] Let $F; \mathbb{R}^2 \to \mathbb{R}^2$ be given by the expression $F(x, y) = (\sin(xy), xy + y)$. Do the following:

(a) Compute the derivative of F.

(b) Along the differential path in $\mathbb{R}^2$ given by $\mathbf{c}(t) = (t^2, 3t - 2)$, write an expression for $\dfrac{dF}{dt}$ and evaluate $\left.\dfrac{dF}{dt}\right|_{t=\frac{2}{3}}$.

Solutions:

(a) $DF = \begin{bmatrix} y\cos(xy) & x\cos(xy) \\ y & x+1 \end{bmatrix}$.

(b) You can do this two ways (although there are the same way, really). One way is the following:

$$\begin{aligned} \frac{dF}{dt} = D(F \circ \mathbf{c})(t) &= DF\left(\mathbf{c}(t)\right) D\mathbf{c}(t) \\ &= \begin{bmatrix} y(t)\cos\left(x(t)y(t)\right) & x(t)\cos\left(x(t)y(t)\right) \\ y(t) & x(t)+1 \end{bmatrix} \begin{bmatrix} 2t \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} (3t-2)\cos\left(t^2(3t-2)\right) & t^2\cos\left(t^2(3t-2)\right) \\ (3t-2) & t^2+1 \end{bmatrix} \begin{bmatrix} 2t \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2t(3t-2)\cos\left(t^2(3t-2)\right) + 3t^2\cos\left(t^2(3t-2)\right) \\ 2t(3t-2) + 3(t^2+1) \end{bmatrix}. \end{aligned}$$

And

$$\left.\frac{dF}{dt}\right|_{t=\frac{2}{3}} = D(F \circ \mathbf{c})\left(\frac{2}{3}\right) = \begin{bmatrix} \frac{4}{3} \\ \frac{13}{3} \end{bmatrix}.$$

The other way is to consider $(F \circ \mathbf{c})$ as just another differentiable path, and differentiate with respect to t component-wise.

Question 2. [20 points] For $F(x,y,z) = y^3 - xz - x$ a real-valued function of three variables and $p = (1,2,7)$ a point in $\mathbb{R}^3$, do the following:

- **(a)** Evaluate ∇F.
- **(b)** Calculate the directional derivative of F at p in the direction $\mathbf{v} = (-2, 1, -2)$.
- **(c)** Find the equation for the tangent plane to the zero-level set of F at p.
- **(d)** Find a point on the zero-level set of F where the tangent plane is parallel to the xy-plane.

Solutions: To save space, I will employ point notation for vectors in $\mathbb{R}^3$ here, so a vector will be written $(\cdot,\cdot,\cdot)$. This is okay since points are considered the same as vectors. What won't work is to use brackets and write the components as a row. That is a row vector and really a different object form a column vector. This is subtle but necessary.

(a) $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(-z-1, 3y^2, -x\right)$.

(b) $D_v F(p) = \nabla F(p) \cdot \frac{\mathbf{v}}{||\mathbf{v}||} = \left(-(7)-1, 3(2)^2, -(1)\right) \cdot \left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{16+12+2}{3} =$ 10**.**

(c) The equation of the tangent plane to F at p is

$\nabla F(p) \cdot (x-1, y-2, z-7) = (-8, 12, -1) \cdot (x-1, y-2, z-7) = 0.$

Thus the equation is

$$-8(x-1) + 12(y-2) - (z-7) = 0 \textbf{ or } -8x + 12y - z = 9.$$

(d) For the tangent plane to any level set of F to be parallel to the xy-plane, ∇F would have to be "vertical", or $(0,0,r)$ for some $r \neq 0$. This means that $y = 0$, $z = -1$, and $x = \pm r \neq 0$ given the answer in (a). So on the zero-level set, we get the equation

$$F(x, 0, -1) = 0$$

which we need to solve for a non-zero x. But this equation is solved for ANY x, since

$$F(x, 0, -1) = 0^3 - x(-1) - x = x - x = 0.$$

Hence choose any point $(x, 0, -1)$ where $x \neq 0$ (this ensures that the normal vector is non-zero) and this point is on the zero-level set and the tangent plane is "horizontal". Two good follow up questions: 1) what happens at the point $(0, 0, -1)$ and 2) what does the zero-level set look like near any point on the line $(x, 0, -1)$?

Question 3. [15 points] Suppose that $w = g\left(\frac{x}{y}, \frac{z}{y}\right)$ is a differentiable function of $u = \frac{x}{y}$ and $v = \frac{z}{y}$. Show that for $\vec{x} = (x, y, z) \in \mathbb{R}^3$, the operator $\vec{x} \cdot \nabla$ vanishes on w. That is, show that

$$\vec{x} \cdot \nabla(w) = x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0.$$

Solution: By the Chain Rule, we get immediately that

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial g}{\partial u}\left(\frac{1}{y}\right) \\
\frac{\partial w}{\partial z} &= \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial z} = \frac{\partial g}{\partial v}\left(\frac{1}{y}\right) \\
\frac{\partial w}{\partial y} &= \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial g}{\partial u}\left(-\frac{x}{y^2}\right) + \frac{\partial g}{\partial v}\left(-\frac{z}{y^2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\vec{x} \cdot \nabla(w) &= x\left(\frac{\partial g}{\partial u}\right)\left(\frac{1}{y}\right) + y\left[\frac{\partial g}{\partial u}\left(-\frac{x}{y^2}\right) + \frac{\partial g}{\partial v}\left(-\frac{z}{y^2}\right)\right] + z\left(\frac{\partial g}{\partial v}\right)\left(\frac{1}{y}\right) \\
&= \left(\frac{x}{y} - \frac{x}{y}\right)\left(\frac{\partial g}{\partial u}\right) + \left(\frac{z}{y} - \frac{z}{y}\right)\left(\frac{\partial g}{\partial v}\right) \\
&= 0.
\end{aligned}$$

Question 4. [15 points] Verify or disprove the following:

(a) There exists a C^2 function $f(x, y, z)$ whose gradient is the vector field

$$\vec{F} = \left(e^x \cos y + e^{-x} \sin z\right) \vec{\mathbf{\imath}} - e^x \sin y \, \vec{\mathbf{\jmath}} - e^{-x} \cos z \, \vec{\mathbf{k}}.$$

(b) There exists a C^2 vector field $\vec{G}(x, y, z)$ whose curl is the vector field

$$\vec{F} = x\left(y^2 + 1\right) \vec{\mathbf{\imath}} + \left(ye^x - e^z\right) \vec{\mathbf{\jmath}} + x^2 e^z \, \vec{\mathbf{k}}.$$

Solutions:

(a) Let $f(x, y, z) = e^x \cos y - e^{-x} \sin z$. Then $\vec{F} = \nabla f$.

(b) If $\vec{F} = \nabla \times \vec{G}$ for some C^2 vector field $\vec{G}$, it would have to be the case that $\nabla \cdot \vec{F} = 0$, since the curl of a vector field is incompressible as a vector field. But

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}\left(x\left(y^2+1\right)\right) + \frac{\partial}{\partial y}\left(ye^x - e^z\right) + \frac{\partial}{\partial z}\left(x^2 e^z\right) \\ &= y^2 + 1 + e^x + x^2 e^z \neq 0. \end{aligned}$$

Hence, it is NOT the case that $\vec{F}$ is the curl of some other vector field $\vec{G}$.

Question 5. [20 points] Use the method of Lagrange multipliers to find the minimum distance from the origin to the surface $x^2 - (y-z)^2 = 1$.

Solution:

The method of Lagrange multipliers leads to the system given by the vector equation $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, and the scalar equation $g(x,y,z) = 1$, where $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ and $g(x,y,z) = x^2 - (y-z)^2$. The system is:

$$\begin{aligned} \frac{x}{\sqrt{x^2+y^2+z^2}} &= \lambda 2x \\ \frac{y}{\sqrt{x^2+y^2+z^2}} &= -\lambda 2(y-z) \\ \frac{z}{\sqrt{x^2+y^2+z^2}} &= \lambda 2(y-z) \\ x^2 - (y-z)^2 &= 1. \end{aligned}$$

Notice that the first equation leads immediately to the multiplier: $\lambda = \dfrac{1}{2\sqrt{x^2+y^2+z^2}}$. With this, the system boils down to

$$\begin{aligned} x &= x \\ y &= -(y-z) \\ z &= y-z \\ x^2 - (y-z)^2 &= 1. \end{aligned}$$

(we could have immediately "chosen" our distance function without the square root sign; This would have made λ "nicer" and the boiled down system would have been the same. You should think about why this works?). The middle two equations are satisfied only for $y = z = 0$ (their graphs are two lines in the yz-plane that cross at the origin), and these values coupled with the last equation and the first yield $x = \pm 1$.

To see that these critical points $(-1,0,0)$ and $(1,0,0)$ must be minima (recall that for functions of more than one variable, one can have more than one local minima without having a local maxima: See Problem 47, p. 259 in the text), look directly at the distance function. For a point along the surface, the x-coordinate must satisfy $x^2 = 1 + (y-z)^2$. Hence

$$f(x,y,z)\Big|_{g(x,y,z)=1} = \sqrt{x^2+y^2+z^2}\Big|_{x^2=1+(y-z)^2} = \sqrt{1+(y-z)^2+y^2+z^2} \geq 1$$

everywhere on the surface. Hence any point of distance 1 from the origin is a minimum.

Question 6. [20 points] A particle moves in $\mathbb{R}^3$ so that its acceleration is a constant $-\mathbf{k}$. If the particle's initial position at $t = 0$ is $(-1, 0, 2)$ and its velocity at $t = 0$ is the vector $\mathbf{i} + \mathbf{j}$.

(a) When does the particle hit the $z = 0$ plane (the floor?)

(b) Where does it hit the floor?

(c) Express the distance the particle travels between $t = 0$ and the moment it hits the floor as an integral of time alone (you do not need to solve the integral).

Solutions:

The acceleration vector is simply the second derivative of the displacement vector, which in this case is the path $\mathbf{c} : [0, \infty) \to \mathbb{R}^3$. Hence $\mathbf{a} = \mathbf{a}(t) = (0, 0, -1) = \mathbf{c}''(t)$. To find c, simply find the second antiderivatives of each of the components separately (this is just Calculus I stuff), and use the initial data to find the constants. Thus $\mathbf{v}(t) = (R_1, R_2, -t + R_3)$ for R_1,R_2, R_3 the three constants of integration. And since $\mathbf{v}(0) = (1, 1, 0)$, we get that $R_1 = R_2 = 1$ and $R_3 = 0$, so that $\mathbf{v}(t) = (1, 1, -t)$. Do this again to get $\mathbf{c}(t) = \left(t - 1, t, -\frac{t^2}{2} + 2\right)$.

(a) Simply solve for the z-component of c to be zero. Hence $0 = -\frac{t^2}{2} + 2$, which is solved for $t = 2$.

(b) $\mathbf{c}(2) = (1, 2, 0)$.

(c) The arclength of c from $t = 0$ to $t = 2$, is

$$\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \int_0^2 \sqrt{2 + t^2}\, dt.$$

One last note: Notice that the number of points here is 20, but on the first page it says that this problem is out of 15 points. I graded this problem out of 20, so with full credit on this problem, you get 5 extra points. The exam is really out of 105 points, even though your final score is marked as a percentage of 100. Hope this is okay. ;-)