Extra Problems: EP3

EP3 Let $X \subset \mathbb{R}^n$ be open. Show $f : X \to \mathbb{R}^m$ is continuous at $\vec{x}_0 \in X$ iff $\lim_{\vec{x}\to\vec{x}_0} ||f(\vec{x}) - f(\vec{x}_0)|| = 0$.

One way is quite easy: Let f be continuous at $\vec{x}_0$. Then $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$, and the result follows.

There are many options for showing the other direction. Perhaps an interesting visual one is the following: Assume that $\lim_{\vec{x}\to\vec{x}_0} ||f(\vec{x}) - f(\vec{x}_0)|| = 0$. Notice that for all $i = 1, \ldots, m$,

$$0 \leq |f_i(\vec{x}) - f_i(\vec{x}_0)| \leq ||f(\vec{x}) - f(\vec{x}_0)|| = \sqrt{\sum_{i=1}^{m} \left(f_i(\vec{x}) - f_i(\vec{x}_0)\right)^2}.$$

This is visual since is it is the statement that the length of a vector in $\mathbb{R}^m$ is longer than the length of any of its coordinates. Equivalently, the length of the diagonal of a parallelepiped is longer than the length of any of its sides (this is a m-dimensional version of a triangle inequality). Each of the elements in the above inequality is a real-valued function of $\vec{x}$, and hence a squeezing lemma argument leads to the fact that if $\lim_{\vec{x}\to\vec{x}_0} ||f(\vec{x}) - f(\vec{x}_0)|| = 0$, then $\lim_{\vec{x}\to\vec{x}_0} |f_i(\vec{x}) - f_i(\vec{x}_0)| = 0$ for all $i = 1, \ldots, m$. But this immediately* implies continuity at $\vec{x}_0$ for each coordinate function of f, given that $\vec{x}_0 \in X$. Since the coordinates functions of f are all continuous at $\vec{x}_0$, f is continuous at $\vec{x}_0$.

To see *, let $g(\vec{x}) = f_i(\vec{x}) - f_i(\vec{x}_0)$. Then since

$$|g(\vec{x})| \geq g(\vec{x}) \geq -|g(\vec{x})|,$$

we get

$$0 = \lim_{\vec{x}\to\vec{x}_0} |g(\vec{x})| \geq \lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) \geq \lim_{\vec{x}\to\vec{x}_0} -|g(\vec{x})| = -\lim_{\vec{x}\to\vec{x}_0} |g(\vec{x})| = 0.$$

Hence

$$\lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) = 0 = \lim_{\vec{x}\to\vec{x}_0} f_i(\vec{x}) - f_i(\vec{x}_0).$$

Finally, since $\lim_{\vec{x}\to\vec{x}_0} f_i(\vec{x}) - f_i(\vec{x}_0)$ and $\lim_{\vec{x}\to\vec{x}_0} f_i(\vec{x}_0)$ each exist, the limit of the sum of $f_i(\vec{x}) - f_i(\vec{x}_0)$ and $f_i(\vec{x}_0)$ exists and is equal to the sum of the limits, so that $\lim_{\vec{x}\to\vec{x}_0} f_i(\vec{x}) = f_i(\vec{x}_0)$.

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