SPRING 2008: 110.211 HONORS MULTIVARIABLE CALCULUS

Extra Problems: EP1, EP2

EP1 Do the following:

- **Show the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is injective if at least one of its component functions is injective.** An easy was to see this is to show the contrapositive: If f is not injective, then all of its components are not injective. To see this, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be two points where $\mathbf{x} \neq \mathbf{y}$, but
$$f(\mathbf{x}) = \mathbf{a} = (a_1, \ldots, a_m) = f(\mathbf{y}).$$
But then for any component $f_i : \mathbb{R}^n \to \mathbb{R}$, $f(\mathbf{x}) = a_i = f(\mathbf{y})$, and hence none of the components are injective. To further explore this idea, think about whether any function from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be injective when $n > m$.
- **Show f is surjective if all of its components are surjective.** This is patently false. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x, x)$. Here, each of its component functions $f_1(x, y, z) = f_2(x, y, x) = x$ is surjective, but the point $(1, 2) \in \mathbb{R}^2$ is not in the image of f. What is true is the converse of the stated problem: If f is surjective, then all of its components are surjective. One proof of this is to assume f is surjective and assume one component is not. You can easily then produce a point not in the image of f, resulting in a contradiction.

EP2 Let $\vec{a}$ and $\vec{b}$ be two vectors in $\mathbb{R}^3$ which are not on the same line, so that the equation $\vec{x} = s\vec{a} + t\vec{b} + \vec{c}$ defines a plane parameterized by the two real variables s, t. Given the non-parameterized equation of the plane $Ax + By + Cz = D$, write the constants A, B, C, D in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

In the parameterized plane, the numbers s and t are the coordinates, the coordinate axes are the lines containing the vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ suitably parallel-transported to the point $\mathbf{c} = (c_1, c_2, c_3)$ (so that the origin of the parameterized plane is at $\mathbf{c}$). Also, the coordinate system on the plane uses $\vec{a}$ and $\vec{b}$ as the unit vectors, no matter their actual lengths.

To find the non-parameterized version of the plane, note that $\vec{n} = \vec{a} \times \vec{b}$ is the normal to the plane, and we will call $\vec{n}_{\mathbf{c}}$ the normal based at $\mathbf{c}$. A point $\mathbf{p} = (p_1, p_2, p_3)$ is in the plane if the displacement vector $\overrightarrow{\mathbf{pc}}$ is perpendicular to $\vec{n}$ (this means $\vec{n}_{\mathbf{c}} \cdot \overrightarrow{\mathbf{cp}} = 0$). Hence we have
$$\vec{n} = (A, B, C), \quad D = Ac_1 + Bc_2 + Cc_3,$$
where
$$A = (a_2b_3 - b_2a_3), \quad B = (a_3b_1 - a_1b_3), \quad \text{and } C = (a_1b_2 - b_1a_2).$$

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