# SPRING 2008: 110.211 HONORS MULTIVARIABLE CALCULUS

Extra Problems: EP6

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**EP6 Let $f$ be continuous on the rectangular region $D = [a,b] \times [c,d]$. For $a \le x \le b$ and $c \le y \le d$, define $F(x,y) = \int_a^x \int_c^y f(u,v)\,dv\,du$. Show that $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = f(x,y)$. This says that Fubini's Theorem for continuous functions implies the equality of mixed partials.**

*Proof.* There is very little to actually do here. By the Fundamental Theorem of Calculus (FTC),

$$\frac{\partial F}{\partial x}(x,y) = \int_c^y f(x,v)\,dv.$$

Apply it again to get $\frac{\partial^2 F}{\partial y \partial x} = f(x,y)$. Use Fubini's Theorem to reverse the order of the iterated integral, and again apply the FTC twice. □

As an exercise to see more directly the relationship between Fubini's Theorem and the mixed partials of a $C^2$ function, assume Fubini's Theorem but neglect the fact that the mixed partials of $F$ are necessarily equal. Let $F(x,y) \in C^2(\mathbb{R}^2, \mathbb{R})$. Fubini's Theorem and the FTC imply

$$\begin{aligned}
\int_a^x \int_c^y \frac{\partial^2 F}{\partial x \partial y}(u,v)\,dv\,du &= \int_c^y \int_a^x \frac{\partial^2 F}{\partial x \partial y}(u,v)\,du\,dv \\
&= \int_c^y \left[\frac{\partial F}{\partial y}(x,v) - \frac{\partial F}{\partial y}(a,v)\right] dv \\
&= F(x,y) - F(x,c) - F(a,y) + F(a,c) \\
&= F(x,y) - F(a,y) - F(x,c) + F(a,c) \\
&= \int_a^x \left[\frac{\partial F}{\partial x}(u,y) - \frac{\partial F}{\partial y}(u,c)\right] du \\
&= \int_a^x \int_c^y \frac{\partial^2 F}{\partial y \partial x}(u,v)\,dv\,du.
\end{aligned}$$

Hence

$$\iint_D \frac{\partial^2 F}{\partial x \partial y}\,dA = \iint_D \frac{\partial^2 F}{\partial y \partial x}\,dA.$$

And since the rectangle $D$ was chosen arbitrarily, it holds for all choices of $D$. Hence the two integrands are equal.

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