EXAMPLE: SECTION 7.1: INTEGRATION BY PARTS (THE ANTI-PRODUCT RULE)

110.109 CALCULUS II (PHYS SCI & ENG)
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Question 1. Calculate $\int \sin^{-1} x \, dx$.

Note. *This is an example of a integral does not have the form of a product of functions. However, by thinking of one of the functions as a constant 1, the Integration by Parts procedure allows us to instead integrate the derivative of the inverse sine function. This will be easier. One other point is that this function* $\sin^{-1} x$ *is NOT "1 over the sine function", or*

$$\sin^{-1} x \neq \frac{1}{\sin x} = \sec x.$$

This is a common mistake due to ambiguous notation. Rather, it is the function-inverse of the sine function, as in $y = \sin^{-1} x$ *is the inverse function of* $\sin y = x$. *To solve this, we will first compute the derivative of* $\sin^{-1} x$. *Then we will use this fact.*

Proof. To compute the derivative of the function $y = \cos^{-1} x$, we will instead use the inverse of this function $\sin y = x$ and use implicit differentiation. Here then thinking of y as an implicit function of x, we have

$$\begin{aligned} \frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) \\ (\cos y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}. \end{aligned}$$

This is not quite complete, as we have only expressed the derivative of $y = \sin^{-1} x$ in terms of the function y. We can fix this through an understanding of how triangles relate the trigonometric functions of their angles to their side lengths and such. Form a right triangle with one angle y and hypotenuse 1. Then the side adjacent to the angle y has length $\cos y$ and the side opposite to the angle y has length $\sin y$. Use the picture at the end as a guide, and notice that the opposite side $\sin y = x$. The Pythagorean's Theorem gives us that $x^2 + \cos^2 y = 1$, or $\cos y = \sqrt{1 - x^2}$. Thus

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}.$$

Why was this helpful? Now use Integration by Parts to rewrite the integral. Here, let $f(x) = \sin^{-1} x$, and $g(x) = 1$ in the formula

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx.$$

Then $f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$, and $g(x) = x$, and we get

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}}\, dx.$$

To finish the calculation, notice that the integral on the right hand side of the last equation can be solved with a straightforward substitution: Let $u = 1 - x^2$, so that $du = -2x\,dx$, or $-\frac{1}{2}\,du = x\,dx$. Then

$$\begin{aligned}
\int \sin^{-1} x\,dx &= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx \\
&= x\sin^{-1} x - \int \frac{-1}{2\sqrt{u}}\,du \\
&= x\sin^{-1} x + \sqrt{u} + C = x\sin^{-1} x + \sqrt{1-x^2} + C.
\end{aligned}$$

$\square$

Is this correct? Let's check by showing that $\frac{d}{dx}\left[x\sin^{-1} x + \sqrt{1-x^2} + C\right] = \sin^{-1} x$. Here

$$\begin{aligned}
\frac{d}{dx}\left[x\sin^{-1} x + \sqrt{1-x^2} + C\right] &= \left(\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}\right) + \frac{1}{2\sqrt{1-x^2}}(-2x) + 0 \\
&= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\
&= \sin^{-1} x.
\end{aligned}$$