1. Invertible $S^1$-maps (cont’d.)

Last class, we established some of the properties of the lifts of circle maps. We continue this now. Again, let $f : S^1 \to S^1$ be a circle map, and $F : \mathbb{R} \to \mathbb{R}$ a lift to $\mathbb{R}$, defined as a map which satisfies the criterion that $f \circ \pi(x) = \pi \circ F(x) \ \forall x \in \mathbb{R}$, where $\pi : \mathbb{R} \to S^1$ is the standard projection given by the exponential map, $\pi(x) = e^{2\pi i x}$, also denoted $\pi(x) = [x]$.

- If $f$ is a homeomorphism, then $|\deg(f)| = 1$.

  Proof. Suppose that $|\deg(f)| > 1$. Then $|F(x + 1) - F(x)| > 1$. And since $F(x + 1) - F(x)$ is continuous, by the Intermediate Value Theorem, $\exists y \in (x, x + 1)$ where $|F(y) - F(x)| = 1$. But then $f([y]) = f([x])$ for some $y \neq x$. Thus $f$ cannot be one-to-one and hence cannot be a homeomorphism.

  Now suppose that $|\deg(f)| = 0$. Then $F(x + 1) = F(x)$, $\forall x$, and hence $F$ is not one-to-one on the interval $(x, x + 1)$. But then neither is $f$, and again, $f$ cannot be a homeomorphism.

- $F(x) - x\deg(f)$ is periodic.

  Proof. It is certainly continuous (why?) To see that it is periodic (of period-1), simply evaluate this function at $x + 1$:

  $$F(x + 1) - (x + 1)\deg(f) = (F(x) + \deg(f)) - (x + 1)\deg(f) = F(x) - x\deg(f).$$

Example 1. Let $f(x) = x$. This is the “identity” map on $S^1$, since all points are taken to themselves. A suitable lift for $f$ is the map $F(x) = x$ on $\mathbb{R}$. to see this, make sure the definition works. Question: Are there any other lifts for $f$? What about the map $F(x) = x + a$ for $a$ a constant? Are there any restrictions on the constant $a$? The answer is yes. For $a$ to be an acceptable constant, we would need the definition of a lift of be satisfied. Thus

$$[F(x)] = [x + a] = f([x]) = [x].$$

So the condition that $a$ must satisfy is $[x + a] = [x]$ on $S^1$. Hence, $a \in \mathbb{Z}$. A new question: For a real number $a \not\in \mathbb{Z}$, can $F(x) = x + a$ serve as a lift of a circle map? What sort of circle map?

Example 2. Let $f(x) = x^n$. Thinking of $x$ as the complex number $x = e^{2\pi i \theta}$, for $\theta \in \mathbb{R}$, then

$$f(x) = f(e^{2\pi i \theta}) = (e^{2\pi i \theta})^n = e^{2\pi i n \theta}.$$  

Hence a suitable lift is obviously $F(x) = nx$ (I say obviously, since the variable in the exponent is the lifted variable!) Question: This is a degree $n$ map. For which values of $n$
does the map $f$ have an inverse" And what does the map $f$ actually do for different values of $n$?

**Example 3.** Let $f$ be a general degree-$r$ map. Then $F(1) - F(0) = r = \text{deg}(f)$. Suppose that $F(0) = 0$. Then $F(1) = r$ and if, for example, $r > 1$, it is now easy to see that there will certainly be a $y \in (0, 1)$, where $F(y) = 1$. This was a fact that we used in the proof above to show that $f$ cannot be a homeomorphism. In this case, where $r > 1$, at every point in $y \in (0, 1)$ where $F(y) \in \mathbb{Z}$, we will have $\pi \circ F(y) = [F(y)] = 0$ on $S^1$. This won’t happen when $r = 1$. When $r = 0$, the map $F$ will be periodic, which is definitely not one-to-one. Question: What happens when $r < 0$? Draw some representative examples to see.

**Definition 4.** Suppose that $f : S^1 \to S^1$ is invertible. Then

1. if $\text{deg}(f) = 1$, $f$ is orientation preserving.
2. if $\text{deg}(f) = -1$, $f$ is orientation reversing.

Recall from Calculus III that orientation is a choice of direction in the parameterization of a space (really, it exists outside of any choice of coordinates on a space, but once you put coordinates on a space, you have essentially chosen an orientation for that space. This is true at least for those spaces that actually are orientable, that is (Moebius Band?) On $\mathbb{R}$, it is the choice of direction for the symbol “>”. On a surface, it is a choice of side. In $\mathbb{R}^3$, one can use the Right Hand Rule. Etc. On $S^1$, orientation preserving really means that after one applies the map, points to the right of a designated point remain on that side. Orientation reversing will flip a very small neighborhood of a point.

Circle maps may or may not have periodic points. And given an arbitrary homeomorphism, without regard to any other specific properties of the map, one would expect that we can construct maps with lots of periodic points of any period. However, it turns out that circle homeomorphisms are quite restricted. because they must be one-to-one and onto, only certain things can happen. To explain, we will need another property of circle homeomorphisms to help us.

**Proposition 5.** Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism, with $F : \mathbb{R} \to \mathbb{R}$ a lift. Then the quantity

$$\rho(F) := \lim_{|n| \to \infty} \frac{F^n(x) - x}{n}$$

(1) exists $\forall x \in \mathbb{R}$,

(2) is independent of the choice of $x$ and is defined up to an additive integer, and

(3) is rational iff $f$ has a periodic point.

Given these qualities, the additional quantity $\rho(f) = [\rho(F)]$ is well-defined and is called the rotation number of $f$. 
Some notes:

- \( \rho \left( R_\alpha \right) = \alpha \). (You should be able to actually calculate this directly using the definition.)
- \( \rho \) represents in a way the average rotation of points in a circle homeomorphism.

**Proposition 6.** If \( \rho(f) = 0 \), then \( f \) has a fixed point.

Another way of saying that if there is no average rotation of the circle map, then somewhere a point doesn’t move under \( f \). This is like the Intermediate value Theorem on a closed, bounded interval of \( \mathbb{R} \) where a map is positive at one end point and negative at the other.

- If \( f \) has a \( q \)-periodic point, then for a lift \( F \), we have \( F^q(x) = x + p \) for some \( p \in \mathbb{Z} \). For example, let \( f = R_\frac{\pi}{7} \). Then a suitable lift for \( f \) would necessarily satisfy \( F^7(x) = x + 6, \forall x \in \mathbb{R} \).

**Proposition 7.** Let \( f : S^1 \rightarrow S^1 \) be an orientation preserving homeomorphism. Then all periodic points must have the same period.

This last point is quite restrictive. Essentially, if an orientation preserving homeomorphism has a fixed point, it cannot have periodic points of any other period, say. Note that this is not true of a orientation reversing map. For example, the map which flips the unit circle in \( \mathbb{R}^2 \) across the \( y \)-axis, will fix the two points \((0, 1)\) and \((0, -1)\), while every other point is of order two.

**Exercise 1.** For \( R_\alpha : S^1 \rightarrow S^1 \) a circle rotation, show \( \rho(R_\alpha) = \alpha \).

**Exercise 2.** Show that any lift of the rotation \( R_\frac{\pi}{7} \) must satisfy \( F^7(x) = x + 6, \forall x \in \mathbb{R} \), and explicitly construct two such lifts.

This is enough for circle homeomorphisms for now. And ends our work in Chapter 4. There is a great section on frequency locking on page 141. Look it over at your leisure. We won’t work through it in the course, but it is very interesting. Dynamically, it represents a situation where a linear flow on the torus (with its uncoupled ODEs) becomes the limiting system to a system of coupled ODEs, representing a nonlinear flow. Question: For this to be the case, must the resulting linear flow on the torus be a rational flow?
2. Chapter 5

In this short chapter, the only thing I want to discuss is a way to understand toral flows and maps in higher dimensions. For this, let’s describe the space. By definition, the \( n \)-dimensional torus, or the \( n \)-torus, denoted \( T^n \) is simply the \( n \)-fold product of \( n \) circles

\[
T^n = S^1 \times \cdots \times S^1.
\]

Think of a system of equations where the \( n \) variables are all angular coordinates. Then

\[
T^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}.
\]

Recall the Kepler Problem. With \( n \) point masses, the resulting flow may be seen as linear motion on \( T^n \).

Another way to view the \( n \)-torus is via an identification within \( \mathbb{R}^n \). Remember the unit square with it opposite sides identified plays a good model for the 2-torus, \( T = T^2 \). The generalization works well here for all the natural numbers. Take the unit cube in \( \mathbb{R}^3 \). Identify each of the opposite pairs (think of a die, and identify two sides if their numbers add up to 7). The resulting model is precisely the \( T^3 \). This works well if one wants to watch a flow on \( T^3 \). Simply allow the flow to progress in the unit cube, and whenever one hits a wall, simply vanish and reappear on the opposite wall, entering back into the cube.

Note this also works well for \( n = 1 \): Take the unit interval and identify its two sides (the numbers \( x = 0 \) and \( x = 1 \)). This is what I mean by the phrase 0 = 1 on \( S^1 \), where the circle is the 1-torus.

Now, the vector exponential map

\[
(\theta_1, \ldots, \theta_n) \xmapsto{\exp} (e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})
\]

maps \( \mathbb{R}^n \) onto \( T^n \). We can define a (vector) rotation on \( T^n \) by the vector \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \), where

\[
R_{\vec{\alpha}} (\vec{x}) = (x_1 + \alpha_1, \ldots, x_n + \alpha_1) = \vec{x} + \vec{\alpha}.
\]

Normally, this is called a translation (by \( \vec{\alpha} \)) on the torus. Note that it should be obvious that if all of the \( \alpha_i \)'s are rational, then the resulting map on \( T^n \) will have closed orbits. The question is, are theses the only periodic linear maps?

Furthermore, consider a flow in \( T^n \) whose time-1 map is \( R_{\vec{\alpha}} \). This would be the constant flow whose \( ith \)-coordinate solution is \( x_i(t) = x_i + \alpha_i t \). Again, with ALL of the \( \alpha_i \)'s rational, the flow would have all closed orbits. Now allow one of the coordinate rotation numbers to be irrational. We saw how it was the ratio of the two flow rates that determined whether the flow had closed orbits on \( T^2 \). Does this hold in higher dimensions? Do the properties of the time-1 map still reflect accurately the properties of the flow? Does the irrationality of some or all of the coordinate rotations imply minimality of the map? of the flow? Really, all of these
questions will rely on a good notion of measuring the relative ratios of the individual pairs of map rotations and flow rates. And how do we define these ratios in higher dimensions? By a notion of the rational independence of sets of numbers:

**Definition 8.** A set of \( n \) real numbers \( \{\alpha_i\}_{i=1}^{n} \) is said to be *rationally independent* if, given \( k_1, \ldots, k_n \in \mathbb{Z} \), the only solution to

\[
k_1\alpha_1 + \ldots + k_n\alpha_n = 0
\]

is for \( k_1 = \cdots = k_n = 0 \).

Another way to say this is the following: For all nontrivial integer vectors \( \vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n - \{\vec{0}\} \),

\[
\sum_{i=1}^{n} k_i\alpha_i = \vec{k} \cdot \vec{\alpha} \neq 0.
\]

We have the following:

**Proposition 9.** A toral translation on \( \mathbb{T}^n \), given by \( R_{\vec{\alpha}} \) is minimal iff the numbers \( \alpha_1, \ldots, \alpha_n, 1 \) are rationally independent.

**Proposition 10.** The flow on \( \mathbb{T}^n \) whose time-1 map is the translation \( R_{\vec{\alpha}} \) is minimal iff the numbers \( \alpha_1, \ldots, \alpha_n, 1 \) are rationally independent.

Do you see the difference? One way to view this is to restrict to the case of a 2-torus. Here, the second proposition says that the flow will be minimal if, in essence, \( k_1\alpha_1 + k_2\alpha_2 = 0 \) is only satisfied when \( k_1 = k_2 = 0 \). Really, if there were another solution, then it would be the case that \( \alpha_2 \alpha_1 = \frac{k_1}{k_2} \in \mathbb{Q} \).

On the other hand, the first proposition indicates that both \( \alpha_1 \) and \( \alpha_2 \) need to be rationally independent and also both rationally independent from 1! That means that not only do the two \( \alpha \)'s need to be rationally independent from each other, but neither \( \alpha_1 \) nor \( \alpha_2 \) can be rational (then it would be a rational multiple of 1). Hence a flow can be minimal on a torus, while the time-1 map isn’t. Why is this so? Let’s study the situation via an example.

**Example 11.** On the two torus, let \( \alpha_1 = \frac{1}{4} \) and \( \alpha_2 = \frac{\pi}{16} \). The flow will be minimal here since \( \frac{\alpha_2}{\alpha_1} = \frac{\pi}{4} \notin \mathbb{Q} \) (\( \alpha_1 \) and \( \alpha_2 \) are rationally independent). However, the time-1 map of this flow is \( R_{\vec{\alpha}} \), and since

\[
k_1 \cdot \frac{1}{4} + k_2 \cdot \pi + k_3 \cdot 1 = 0 \text{ is solved by } [k_1 \ k_2 \ k_3] = [4 \ 0 \ 1],
\]

the translation will not be minimal (the orbits are not dense in the torus). The fact that \( \alpha_1 \) is already rational is the problem. The figure will tell the story. Essentially, the orbit coordinates of the translation in the \( x_1 \) direction will only take the values \( 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \), while the \( x_2 \)-coordinates will “fill out” the vertical direction. The result is that the orbit of the
translation will only be dense on the vertical lines corresponding to the \( x_1 \)-coordinates of the orbit. This sits in contrast to the flow, which will every orbit will “fill” the torus.