110.109 CALCULUS II

Week 8 Lecture Notes: March 28 - April 1

Lecture 1: Section 10.2 Series

Today we start the discussion on a series, basically the sum of the terms of a sequence:

Definition 1. The sum of all of the terms in a sequence $\{a_n\}$ is called an (infinite) series, and denoted

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \quad \text{or simply} \quad \sum a_n.$$

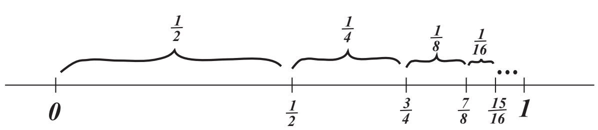
Note that sometimes this sum doesn't exist:

$$a_n = n, \quad \text{so that} \quad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots = \infty,$$

and sometimes it does:

$$a_n = \frac{1}{2^n}. \quad \text{Here} \quad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1??$$

To see geometrically this "sum", look on the number line. Adding $\frac{1}{2}$ to 0 and then successively adding half of the previous amount in turn, notice how you will never exceed 1. Yet if you believe that you will finish somewhere before 1, convince yourself that you will eventually pass that number. This looks like a limit-type process and you are correct. The sum is indeed 1, but to see this will take a bit of work.



First, denote by s_n the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$, so that

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

As we push n toward ∞, these partial sums will tend toward a total sum, at least when the sum actually exists.

Example 2. For $a_n = \frac{1}{2^n}$, we have

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \quad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \quad \cdots \quad .$$

Do you sense a pattern? We have

$$s_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}.$$

For a given series, the set of all partial sums forms a new sequence $\{s_n\}_{n=1}^{\infty}$. We can say that where this sequence goes, so will go the series.

Definition 3. Given a series $\sum_{n=1}^{\infty} a_n$ with its associated sequence of partial sums $\{s_n\}$, if $\lim\limits_{n\to\infty} s_n = s$ exists (so that $\{s_n\}$ is convergent), then the series is convergent and

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = s.$$

We call s the sum of the series, if it exists. If the sum does not exist, we say that the series is divergent.

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Example 4. Again, back to $a_n = \frac{1}{2^n}$. we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n},$$

as long as the two limits on the right hand side exist. They do, and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 = 0 = 1.$$

The series is convergent and its sum is 1.

Definition 5. A series of the form

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \not= 0,$$

is called a *geometric* series.

In a geometric series, each term is the product of the previous term and the factor r, so that the terms of the series look like $a_{n+1} = ra_n$ for all $n \in \mathbb{N}$. As an example, the previous series we have been discussing, namely

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^{n-1}$$

is geometric with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

A huge question involving a series in general, and a geometric series in particular, is: Does the series converge? To answer this, we need to appeal to the sequence of partial sums. Let's look specifically at how the value of r determines whether the geometric series converges or not.

Case 1. *Let* $r = 1$. Then for $a \neq 0$ (there is not much to discuss in the case that $a = 0$, no?) the series $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a$. The partial sum is then

$$s_n = a + a + \cdots + a = na,$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} na = a \lim_{n\to\infty} n = \infty.$$

Hence the series diverges.

Case 2. *Let* $r \neq 1$. Here then the series has its partial sum

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

This is not so easy to manage, but there are clever ways to play with this expression. Here is one; note that

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Then

$$s_n - rs_n = a - ar^n,$$

since all of the other terms cancl out (this is why the difference here is clever!). Now, solve for s_n to get $s_n(1 - r) = a(1 - r^n)$, or

$$s_n = \frac{a(1 - r^n)}{1 - r}.$$

So does the series with this partial sum sequence converge? we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} 1-r^n,$$

which will converge precisely when $|r| < 1$, and diverge precisely when $|r| \geq 1$.

Note: As a special case, let $r = -1$. Then $\lim\limits_{n\to\infty} 1 - r^n$ does not converge, but not because the sequence of partial sums gets large. Instead, here,

$$s_n = \frac{a(1-r^n)}{1-r} = \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

The sequence of partial sums does not exist here, even though the sequence is bounded. Hence it is still the case that the geometric series diverges for $r = -1$.

Lecture 2: Section 10.2 Series (cont'd.)

Here are some examples of geometric series that you have already played with.

Example 6. *Exponential Functions.* Let $f(x) = b^x$, where $b > 0$. Then we can create a sequence $a_n = f(n) = b^n$. This sequence is geometric, with the initial constant $a = b$, and the base $r = b$, and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} b\,(b)^{n-1}.$$

Again, this series will converge iff $b < 1$. Thus, there is a close relationship between exponential functions and geometric series. we say that exponential function are examples of *geometric growth.*

Exercise 1. Find a value for b so that the series $\sum\limits_{n=1}^{\infty} b_n = 5$.

Solution. While this may seem straightforward, there is one thing to consider. To use the sum formula for a geometric series, it is absolutely necessary to correctly identify the values for a as well as r: Here again

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} b\,(b)^{n-1},$$

so that $a = r = b$. Now we can use the folmula

$$\sum_{n=1}^{\infty} b\,(b)^{n-1} = \frac{b}{1-b} = 5.$$

The last equation is easy to solve with a little algebra, and we find $b = \frac{5}{6}$ (check this!).

Example 7. *Decimal representations are geometric series.* the rational number $\frac{5}{9}$ has the decimal representation $.55555\overline{5}$, where the bar over the last 5 indicates that the pattern continues. But recall the definition of a decimal representation means that

$$.55555\overline{5} = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \cdots.$$

This is simply a geometric sequence with $a = \frac{5}{10}$ and $r = \frac{1}{10}$, and

$$.55555\overline{5} = \sum_{n=1}^{\infty} \frac{5}{10}\left(\frac{1}{10}\right)^{n-1}.$$

Notice again that this geometric sequence converges since $|r| = \left|\frac{1}{10}\right| < 1$. And it converges to the quantity $\frac{a}{1-r}$, which in this case is

$$\sum_{n=1}^{\infty} \frac{5}{10}\left(\frac{1}{10}\right)^{n-1} = \frac{a}{1-r} = \frac{\frac{5}{10}}{1-\frac{1}{10}} = \frac{\frac{5}{10}}{\frac{9}{10}} = \frac{5}{9},$$

as it should.

Example 8. *Determine whether the series* $\sum_{n=1}^{\infty} 5(2^{n+2})(3^{-n+1})$ *converges. If it does, then find the sum.* Due to the presence of the index variable n in the exponent, and not anywhere else, this series has a good chance of being a geometric series, and if we can put it into the form of a geometric series (by correctly identifying both a and r), then perhaps we can answer the question. To this end, let's try to manipulate the term $a_n = 5(2^{n+2})(3^{-n+1})$. Here we have

$$a_n = 5(2^{n+2})(3^{-n+1}) = 5 \cdot 2^3 \cdot 2^{n-1} \cdot \frac{1}{3^{n-1}} = 40 \cdot \left(\frac{2}{3}\right)^{n=1}.$$

Notice that we have stripped out a 2^3 from 2^{n+2}, knowing that $2^3 2^{n-1} = 2^{3+(n-1)} = 2^{n+2}$, and $3^{-n+1} = 3^{-(n-1)} = \left(\frac{1}{3}\right)^{n-1}$. Putting these together, we wind up with a term that meets the form of a geometric series, with $a = 40$ and $r = \frac{2}{3}$. This series converges since $|r| < 1$, and the sum is

$$\sum_{n=1}^{\infty} 5(2^{n+2})(3^{-n+1}) = \sum_{n=1}^{\infty} 40\left(\frac{2}{3}\right)^{n-1} = \frac{40}{1-\frac{2}{3}} = 120.$$

Now if the series is NOT geometric, then we cannot use the convergence criterion used above. We can still appeal to the original notion that if we can find a good expression for the terms of the sequence of partial sums s_n, we cam try to determine if the sequence has a limit. If it does, then the series also converges and to the limit of the sequence of partial sums.

Outside of this, sometimes one needs to simply be clever and look for patterns and structure to exploit to see if a given series may converge or not. Examples 6 and 7 in the text are two good examples of this type of cleverness. example 6 deals with a series called a telescoping series, where each succeeding pair of terms of the series almost cancel each other out, leaving very little but a couple of terms in the partial sum. This gives a useful expression for the partial sum and the convergence of the series is determined by analyzing the sequence of partial sums.

In example 7, they study the Harmonic Series. Let's do that one in detail:

Example 9. *The Harmonic Series.* We saw that the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converged, even though it was a sum of an infinite number of positive numbers. The point was that the positive numbers were decreasing quickly and quickly enough that the ultimate sums didn't exceed (in this case) 1. Here, we introduce the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

Here, the terms of the sequence are all still positive numbers which form a decreasing sequence, but they decrease much more slowly. Also, there is no easy way to write out a useable expression for the partial sum

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

And the sequence is certainly not geometric.

It turns out that the series diverges, and in this case, that means that the sum is infinite. How to show this involves showing that the sequence of partial sums eventually exceeds every integer. Indeed, we will do this by limiting ourselves to only certain of the partial sums, and noting a useful pattern.

To start, notice that the first two partial sums have a pattern to them, if seen in a certain way:

$$s_1 = 1 = 1 + \frac{0}{2}, \quad \text{and} \quad s_2 = 1 + \frac{1}{2}.$$

I like the pattern, and notice that if we skip to s_4, we get

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} = 2$$

Really, this says nothing more than the fact that $s_4 > 2$.

But there is a pattern here: If we pass to s_8, we notice that

$$s_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

which means that $s_8 > 2.5$. We can continue this pattern by next looking at s_{16} and noting that it is strictly larger than $1 + \frac{4}{2} = 3$, then to s_{32}, noting that it is strictly larger than $1 + \frac{5}{2}$, and generalizing to the pattern that at the partial sum of the 2^nth term, we have

$$s_{2^n} > 1 + \frac{n}{2}.$$

Do you see how this is helpful. Say I wanted to know when the partial sum of the harmonic series pass the sum of 100. Then I simply solve $100 = 1 + \frac{n}{2}$, This give me $n = 198$. So then I know that the partial sum

$$s_{2^{198}} > 100.$$

Since I can do this for every number n, eventually, my partial sum of the harmonic series will pass every integer. This means that $\lim_{n\to\infty} s_n = \infty$, so that the harmonic series *diverges.*

By the way, do not worry about having to be so clever in finding ways to show that a series converges or not. In time, you will develop such abilities. For now, know that there are many useful tools to determine whether a series converges or not. The rest of this section and the next few will develop some of those ways. Here is one:

Theorem 10. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

All this theorem really says is that the only way that a series can converge is if the sequence of partial sums has a limit. That sequence of partial sums can only have a limit if the sequence tends to a number . That means that the things we add to each partial sum to get the next one need to decay away to 0 as the index n goes to infinity. Makes sense, but be very careful here:

CAUTION. The converse of this statement is NOT true!

The statement: "If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges" is false! The harmonic series is one example where the terms go to 0, but the series diverges. Be very careful here.

Instead, the contrapositive of the theorem is true. The contrapositive of a conditional statement is formed by negating both antecedent (the "if part) and the consequent (and "and" part), and switching them. We get:

Theorem 11. *If* $\lim_{n\to\infty} a_n \neq 0$ *(or does not exist), then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

One can call this The Divergence Test for a series. It is very effective.

Example 12. *Show* $\sum_{n=1}^{\infty} \frac{1+n^2}{5n^2+6n}$ *diverges.* Here, we see

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{5n^2+6n} = \frac{1}{5} \neq 0.$$

At this point, we know by the Divergence Test that the series will diverge.

Other conclusions to mention? Well, much like limits, series behave well in sums, differences, and the like (after all, there ARE defined by limits of their partial sums).

Suppose $\sum a_n$ and $\sum b_n$ are two convergent series (this only works when these sums are convergent). Then

(1) $\sum ca_n = c\sum a_n$ converges, and
(2) $\sum a_n \pm b_n = \sum a_n \pm \sum b_n$ also converges.

Here, the symbol $\pm$ means that when there is a "plus" on the left-hand side, there is also a plus on the right, and when there is a "minus" on the left, three is also one on the right. This should all make sense, but again, only if the individual sums exist. There is no need to prove this result. Just keep it in mind.

next time, we will do an example.

Lecture 3

we start today with an example of the final result from last lecture.

Example 13. *Find the sum of* $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)} \right)$, *if it exists.* Here, trying to combine the two terms in the sum will not create an easier fraction to study. However, with the previous result, we can try the following: See if, when we rewrite the series as

$$\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)} \right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)},$$

each of the two separate series on the right converge. If they do, then the original series will converge also, and to the sum of the two individual series on the right. Only if the two on the right converge will this work, but it is worth the effort.

Let's take each of the series on the right separately. First, notice that the series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ looks geometric, due to the fact that the only place we find the n is in the exponent. When we rewrite it:

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \frac{1}{e} \left(\frac{1}{e}\right)^{n-1},$$

we find that it is indeed geometric, with $a = \frac{1}{e}$, and $r = \frac{1}{e}$. Noting that this choice of r satisfies $|r| < 1$, we know that this series converges and the sum is $\frac{a}{1-r}$, so that

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \frac{1}{e} \left(\frac{1}{e}\right)^{n-1} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}.$$

For the second series, try to find a good expression for the partial sum

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)}.$$

Good luck with that. Outside of this, try to use any structure found in the terms to possibly "uncover" something you can work with. for example, noting that this rational expression of n can be decomposed into two simpler ones via a partial fraction decomposition, we get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

While this might not look like much help, go back to the partial sum:

$$\begin{aligned} s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

since all of the intermediate terms cancel out. This is called a *telescoping* sequence, since each succeeding term will kill off all or part of a preceding term. Hence we do have a good expression for the partial sum. So here $s = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, if it exists, where

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1.$$

Hence the second series also converges, and we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{e-1} + 1 = \frac{e}{e-1}.$$

So here is a new question: We know that the series $\sum \frac{1}{n}$ diverges, yet $\sum \frac{1}{n(n+1)} = \sum \frac{1}{n^2+n} = \sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges to 1.

Question 14. *How about* $\sum_{n=1}^{\infty} \frac{1}{n^2}$*?*

There is no simple expression for the partial sum

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} \\ &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2}. \end{aligned}$$

But notice the following:

The (unbounded) area between the curve of $f(x) = \frac{1}{x^2}$ and the x-axis on the interval $[1, \infty)$ is

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to\infty} 1 - \frac{1}{b} = 1.$$

Hence the area of the region outlined in red is

$$1 + \int_1^\infty \frac{1}{x^2}\,dx = 2.$$

The block shaded region is a geometric representation of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The total area of the infinite number of blocks of length 1 and height $\frac{1}{n^2}$, is precisely the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Notice that it is strictly less than the area of the block red region. Hence we must have that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$$

This means that the series will converge. Why is this so. First, since the entire series is less than 2, each partial sum will be less than 2. Hence the sequence of partial sums will be bounded by 2. Note that the sequence of partial sums will also be a monotonic sequence (always adding a positive number each time. Hence it must converge. Thus the series will converge. Now we do not know what it will converge to here (actually, we do, and the sum is $\frac{\pi^2}{6}$, but there is no way here to know that here), but just knowing that it does converge is progress. And we found out by comparing to the improper integral of the function which, in some way, generated the sequence.

We can play the same game with the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$. Here is another figure. To study this series, create the function $f(x) = \frac{1}{x}$ so that our series is the sum of all of the terms $a_n = f(n)$ from 1 to ∞. By graphically representing the series as the sum of the areas of the blocks here, we note that

$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^\infty \frac{1}{x}\,dx.$$

In this case, the right hand side IS ∞ (the integral diverges). But then this means that the harmonic series will also diverge (the sum will also be infinity) since it will be strictly larger than something that does not converge. Again, we studied the series by instead comparing it to an improper integral. Where the integral went, so went the series. We have the following:

Theorem 15. *Suppose $f(x)$ is a positive, continuous, and decreasing function on $[1, \infty)$, and $a_n = f(n)$. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges iff } \int_1^\infty f(x)\,dx \text{ converges.}$$

Some notes:

- This is called the *Integral Test for the Convergence of a Series.*
- This test is great for studying series whose terms look like a function one can integrate.
- This test does not give the sum. While it is tempting to think the improper integral should also give the sum, it does NOT in general.
- Really, where a series starts is not important, and this test works also for series that start later than 1. See the next example.

Example 16. *We now know that* $\sum_{n=2}^{\infty} \frac{1}{n^2}$ *converges, while* $\sum_{n=2}^{\infty} \frac{1}{n}$ *diverges. What about* $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$*?* This one seems to be in between the two. Fortunately, we can use the Integral Test here: Let $f(x) = \frac{1}{x \ln x}$. The improper integral of $f(x)$ is

$$\int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du.$$

The antiderivative of $\frac{1}{u}$ on the interval $[\ln 2, \infty)$ is $\ln u_+ C$. Hence the antiderivative of $f(x) = \frac{1}{a \ln x}$ on the interval $[2, \infty)$ is $\ln \ln x$. And does this antiderivative have a horizontal asymptote? You should work all of this out, but the answer is no. Hence the integral diverges. Hence also the series diverges.

Question 17. *You know that* $\sum_{n=2}^{\infty} \frac{1}{n^2}$ *converges. What about* $\sum_{n=2}^{\infty} \frac{1}{n^p}$*, when* $p > 2$*?*

Question 18. *Show that* $\sum_{n=2}^{\infty} \frac{1}{e^{n^2}}$ *converges by the integral test. Hint: You cannot do the resulting improper integral directly, but you can use the Comparison test for Improper Integrals to help you. See previous lecture notes.*