

110.109 CALCULUS II

Week 7 Lecture Notes: March 14 - March 18

IMPROPER INTEGRALS (CONT'D.)

Go back to the example of an integral where the integrand is $f(x) = \frac{1}{x^2}$, but this time choose to integrate on the interval $[b, 1]$, where $0 < b \leq 1$. For a choice of b here, we get

$$\int_b^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_b^1 = -1 + \frac{1}{b} > 0.$$

We know it is greater than 0 since the entire graph of $f(x) > 0$, and also for the last expression $\frac{1}{b} > 1$ (why?)

Question 1. *What if we pushed b all the way back to 0?*

Again, we would produce a problem with the limits of the integral. And again, the problem would involve an interpretation of the definite integral as the area of an unbounded region. However, this time, the region is unbounded in the vertical direction due to the vertical asymptote at $x = 0$. To deal with this situation, we employ the same trick as before, noting that the definite integral is perfectly well-defined for all positive values of b : define

$$\int_0^1 \frac{1}{x^2} dx = \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{x^2} dx = \lim_{b \rightarrow 0^+} \left(-\frac{1}{x} \Big|_b^1 \right) = \lim_{b \rightarrow 0^+} \left(-1 + \frac{1}{b} \right) = \lim_{b \rightarrow 0^+} \left(\frac{1-b}{b} \right).$$

Finishing this calculation means evaluating the limit at the end of the equations above. We will get ∞ , and can interpret this as the area between $f(x)$ and the x -axis grows without bound as we push the lower limit back to 0. This is an example of another type of improper integral, where the integrand is not defined at one or both of the limits.

Example 2. *Do the same thing by replacing the integrand with the new function $g(x) = \frac{1}{\sqrt[3]{x}}$.* Note that the graph look remarkably similar, and there is again a vertical asymptote of $g(x)$ at $x = 0$. However, this time, we get

$$\int_0^1 \frac{1}{\sqrt[3]{x}} dx = \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{\sqrt[3]{x}} dx = \lim_{b \rightarrow 0^+} \left(\frac{3}{2} x^{\frac{2}{3}} \Big|_b^1 \right) = \lim_{b \rightarrow 0^+} \left(\frac{3}{2} - \frac{3}{2} \sqrt[3]{b^2} \right) = \frac{3}{2}.$$

Here the resulting limit does exist (the function $\sqrt[3]{x^2}$ is continuous from the right at $x = 0$). The interpretation is that the area of the unbounded region between the curve and the x -axis is finite and is $\frac{3}{2}$.

Example 3. *How about $h(x) = \frac{(x+1)(x-2)}{x-2}$ on the interval $[1, 3]$?* Here both limits of the integral of $h(x)$ would be fine. However, there is a point *inside* the interval (namely $x = 2$) where the function is not defined. We cannot simply ignore this point. We can, however, adjust the calculation to accommodate it knowing the Sum Law for Integrals (Section 5.2, page 374). We can write

$$\int_1^3 \frac{(x+1)(x-2)}{x-2} dx = \int_1^2 \frac{(x+1)(x-2)}{x-2} dx + \int_2^3 \frac{(x+1)(x-2)}{x-2} dx,$$

knowing that both of the integrals on the right-hand side are improper. Keep in mind that they are improper because one of their limits is outside of the domain of the function, and not because there is an asymptote

there (there isn't in this case). In this calculation, we get

$$\begin{aligned}\int_1^3 \frac{(x+1)(x-2)}{x-2} dx &= \int_1^2 \frac{(x+1)(x-2)}{x-2} dx + \int_2^3 \frac{(x+1)(x-2)}{x-2} dx \\ &= \lim_{c \rightarrow 2^-} \int_1^c \frac{(x+1)(x-2)}{x-2} dx + \lim_{c \rightarrow 2^+} \int_c^3 \frac{(x+1)(x-2)}{x-2} dx \\ &= \lim_{c \rightarrow 2^-} \int_1^c (x+1) dx + \lim_{c \rightarrow 2^+} \int_c^3 (x+1) dx.\end{aligned}$$

Notice here two things: (1) We set up the limits only from one side. In each case, we only need to evaluate the limit from the side *within* the interval of integration. This is very important. And (2), since we are now off the “bad” point at $x = 2$, the function $h(x)$ is equivalent to $x + 1$, and we can simplify the integrand. Continuing the calculation

$$\begin{aligned}\int_1^3 \frac{(x+1)(x-2)}{x-2} dx &= \lim_{c \rightarrow 2^-} \int_1^c (x+1) dx + \lim_{c \rightarrow 2^+} \int_c^3 (x+1) dx \\ &= \lim_{c \rightarrow 2^-} \left(\frac{x^2}{2} + x \Big|_1^c \right) + \lim_{c \rightarrow 2^+} \left(\frac{x^2}{2} + x \Big|_c^3 \right) \\ &= \lim_{c \rightarrow 2^-} \left(\frac{c^2 + 2c}{2} - \frac{3}{2} \right) + \lim_{c \rightarrow 2^+} \left(\frac{15}{2} - \frac{c^2 + 2c}{2} \right).\end{aligned}$$

Both of these limits exist as the expressions are perfectly continuous at $x = 2$. To finish,

$$\begin{aligned}\int_1^3 \frac{(x+1)(x-2)}{x-2} dx &= \lim_{c \rightarrow 2^-} \left(\frac{c^2 + 2c}{2} - \frac{3}{2} \right) + \lim_{c \rightarrow 2^+} \left(\frac{15}{2} - \frac{c^2 + 2c}{2} \right) \\ &= \left(\frac{8}{2} - \frac{3}{2} \right) + \left(\frac{15}{2} - \frac{8}{2} \right) = \frac{12}{2} = 6.\end{aligned}$$

Hence the improper integral exists and its value is 6.

So let's now define exactly what I mean by an improper (definite) integral of this kind, and the plan for evaluating it, if possible.

Definition 4. If $f(x)$ is continuous on $[a, b)$ and discontinuous at $x = b$, then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx,$$

provided the limit exists.

Definition 5. If $f(x)$ is continuous on $(a, b]$ and discontinuous at $x = a$, then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx,$$

provided the limit exists.

Note that, like before, if the limit exists in either case, we say that the improper integral *converges*. If the limit does not exist, then the integral *diverges*.

Definition 6. If $f(x)$ is continuous on $[a, b]$ except at the point $a < c < b$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx,$$

where at least one of the integrals on the right-hand side are improper.

Example 7. Calculate $\int_0^{2\pi} f(x) dx$, if possible, for the piecewise-defined function

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, \pi) \\ \cos(x - \pi) & \text{if } x \in [\pi, \infty) \end{cases}.$$

Here we note that the function is discontinuous at $x = \pi$, and use the last definition to write

$$\begin{aligned} \int_0^{2\pi} f(x) dx &= \int_0^{\pi} x dx + \int_{\pi}^{2\pi} \cos(x - \pi) dx \\ &= \lim_{c \rightarrow \pi^-} \int_0^c x dx + \int_{\pi}^{2\pi} \cos(x - \pi) dx \end{aligned}$$

since only one of the integrals is actually improper ($f(x)$ is continuous from the right at $x = \pi$. Hence the last integral on the right is okay). Continuing, we get

$$\begin{aligned} \int_0^{2\pi} f(x) dx &= \lim_{c \rightarrow \pi^-} \int_0^c x dx + \int_{\pi}^{2\pi} \cos(x - \pi) dx \\ &= \lim_{c \rightarrow \pi^-} \left(\frac{x^2}{2} \Big|_0^c \right) + \left(\sin(x - \pi) \Big|_{\pi}^{2\pi} \right) \\ &= \lim_{c \rightarrow \pi^-} \left(\frac{c^2}{2} \right) = \frac{\pi^2}{2}. \end{aligned}$$

Remark 8. You may have noticed (like in Examples 3 and 7) that when there is no asymptote involved, that the limit exists and hence the improper integral converges. This is NOT always the case, and it is always necessary to evaluate the limit explicitly than to make an assumption.

Example 9. Calculate $\int_0^3 g(x) dx$, if possible, for the piecewise-defined function

$$g(x) = \begin{cases} 1 & \text{if } x \leq 2 \\ \frac{1}{(x-2)^2} & \text{if } x > 2 \end{cases}.$$

Note that here, as in the last example, the domain of $g(x)$ is all real numbers. However, there is a vertical asymptote for $g(x)$ at $x = 2$.

We start by separating out the discontinuity:

$$\begin{aligned} \int_0^3 g(x) dx &= \int_0^2 dx + \int_2^3 \frac{1}{(x-2)^2} dx \\ &= \int_0^2 dx + \lim_{c \rightarrow 2^+} \int_c^3 \frac{1}{(x-2)^2} dx \end{aligned}$$

since $g(x)$ is continuous from the right at $x = 2$. Continuing, we get

$$\begin{aligned} \int_0^3 g(x) dx &= \int_0^2 dx + \lim_{c \rightarrow 2^+} \int_c^3 \frac{1}{(x-2)^2} dx \\ &= \left(x \Big|_0^2 \right) + \lim_{c \rightarrow 2^+} \left(-\frac{1}{(x-2)} \Big|_c^3 \right) \\ &= 2 + \lim_{c \rightarrow 2^+} \left(-1 + \frac{1}{c-2} \right) = 2 + \lim_{c \rightarrow 2^+} \left(\frac{3-c}{c-2} \right) = \infty. \end{aligned}$$

This integral diverges even though part of it is fine.

Example 10. For $a > 0$, calculate $\int_{-a}^a \frac{1}{\sqrt[5]{x}} dx$, if possible. This integrand is an odd function. Hence it is symmetric with respect to the origin. Recall the the integral of an odd continuous function on the interval $[-a, a]$ should be 0. However, since the integral is improper (asymptote at $x = 0$), we do not know if the integral even converges. To check, we need to actually do the calculation.

Here we get

$$\begin{aligned} \int_{-a}^a \frac{1}{\sqrt[5]{x}} dx &= \int_{-a}^0 \frac{1}{\sqrt[5]{x}} dx + \int_0^a \frac{1}{\sqrt[5]{x}} dx \\ &= \lim_{c \rightarrow 0^-} \int_{-a}^c \frac{1}{\sqrt[5]{x}} dx + \lim_{c \rightarrow 0^+} \int_c^a \frac{1}{\sqrt[5]{x}} dx \\ &= \lim_{c \rightarrow 0^-} \left(\frac{5}{4} \sqrt[5]{x^4} \Big|_{-a}^c \right) + \lim_{c \rightarrow 0^+} \left(\frac{5}{4} \sqrt[5]{x^4} \Big|_c^a \right) \\ &= \lim_{c \rightarrow 0^-} \left(\frac{5}{4} \sqrt[5]{c^4} - \frac{5}{4} \sqrt[5]{(-a)^4} \right) + \lim_{c \rightarrow 0^+} \left(\frac{5}{4} \sqrt[5]{a^4} - \frac{5}{4} \sqrt[5]{c^4} \right) \\ &= -\frac{5}{4} \sqrt[5]{a^4} + \frac{5}{4} \sqrt[5]{a^4} = 0. \end{aligned}$$

The thing here is that both of the limits actually do exist, and are the same magnitude yet opposite signs. Hence the origin integral converges for any value of $a > 0$, and the integral is 0.

LECTURE 2: IMPROPER INTEGRALS (CONT'D) AND SEQUENCES

Finding the value of a definite integral which is improper means evaluating a limit as well as using the Fundamental Theorem of calculus (using the antiderivative evaluated at the limits). However, many time, the actual value of the integral is not nearly as important as knowing whether the integral converges or not. This is particularly true when the integral is difficult or impossible to solve analytically. Fortunately, and like other areas of calculus, integrals behave well when compared to each other. We start with a major theorem:

Theorem 11. Suppose $f(x)$, $g(x)$ are continuous with $f(x) \geq g(x) \geq 0$ for all $x \geq a$. Then if $\int_a^\infty f(x) dx$ converges, then $\int_a^\infty g(x) dx$ converges. And if $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ diverges.

Notes:

- The improper integrals above represent unbounded areas between their respective curves and the x -axis. The integral of the larger function must be bigger than that of the smaller (there is more area). Thus if the integral of the larger is finite, the area of the smaller must also be finite. Also, if the area of the smaller is not finite, then neither can the area of the larger.
- This is also true is the two continuous functions cross each other a number of times, as long as at some point they stop crossing and one dominates the other in the “tail”.
- really, this is ONLY an existence theorem, and says nothing about the actual value of either of the integrals, if they exist.

I then did example 9 from this section in detail. Also pay attention to Example 10.

On to sequences.

Definition 12. A *sequence* of numbers is simply a infinite list of numbers, denoted by a variable and a subscript that takes values in the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$,

$$a_1, a_2, \dots, \quad \text{or} \quad \{a_n\} \quad \text{or} \quad \{a_n\}_{n \in \mathbb{N}} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

Notes:

- Sometimes, the subscript will take values other than the natural numbers. For example, it may start at 0, or may incorporate some of the negative integers. it will be obvious, however, from the context, what the range of values of the subscript will be in the application.
- Many times, the sequence follows a pattern which may be set out according to a rule (like a function) based on the subscript:

$$a_n = \frac{1}{n}, \quad \text{or} \quad b_n = \frac{(-1)^n}{2^n}, \quad \text{or} \quad c_n = \sin 2\pi n.$$

These sequences are actually functions of their subscript, and all can be written as $a_n = f(n)$. The first and last look like the integer points of a continuous function. For example, $a_n = f(n)$, where $f(x) = \frac{1}{x}$. This would be harder to do for the sequence $\{b_n\}$.

- Graphing a sequence is like graphing $y = f(x)$, except that the graph is a discrete set of points in the plane with coordinates (n, a_n) .

Definition 13. A sequence $\{a_n\}$ has a limit L and we write

$$\lim_{n \rightarrow \infty} \{a_n\} = L, \quad \text{or} \quad \lim_{n \rightarrow \infty} a_n = L, \quad \text{or} \quad \{a_n\} \rightarrow L$$

if, for every $\epsilon > 0$, there is a corresponding N where, if $n > N$, then $|a_n - L| < \epsilon$.

Here, the last inequality is not difficult to see. $|a_n - L| < \epsilon$ is the same as $-\epsilon < a_n - L < \epsilon$, which is the same as $L - \epsilon < a_n < L + \epsilon$. This is simply a small band around a number L of width ϵ in either direction (for a total width of 2ϵ). The definition says that given a sequence, the number L will be the limit of the sequence provide no matter how small we make the band, eventually the sequence will enter that band and never leave again. This is exactly the same as what you studied for function way back in Calculus I.

Really, having a limit or not is the ONLY interesting thing about sequences. And it is an important quality to have. So having the means to determine if a sequence has a limit and to be able to find it if it does have one are paramount. There are many ways to discover limits of sequences.

Use the function. Suppose $a_n = f(n)$ for some function $f(x)$ which is continuous on $[1, \infty)$. If $f(x)$ has a horizontal asymptote at ∞ , then the sequence has no choice but to follow the function, and the sequence converges. In other words:

Proposition 14. For $a_n = f(n)$, where $f(x)$ is continuous on $[1, \infty)$, if $\lim_{x \rightarrow \infty} f(x) = L$, then $\lim_{n \rightarrow \infty} a_n = L$.

Example 15. Let $a_n = 2 - \frac{1}{n}$. Since $f(x) = 2 - \frac{1}{x} = \frac{2x-1}{x}$ is continuous on $[1, \infty)$, and $\lim_{x \rightarrow \infty} f(x) = 2$, it follows that $\lim_{n \rightarrow \infty} a_n = 2$ also.

Note: The converse is definitely NOT true. Just because a sequence defined by $a_n = f(n)$ converges, it does not follow that the original function has a horizontal asymptote at infinity!

Example 16. Let $f(x) = \cos 2\pi x$. $f(x)$ is continuous for all reals. And $f(x)$ does not have a horizontal asymptote at infinity. This means that $\lim_{x \rightarrow \infty} f(x)$ does not exist. It is the cosine function. But

$$a_n = f(n) = \cos 2\pi n = 1$$

for all $n \in \mathbb{N}$. This sequence does converge (to 1, that is). In fact, the sequence is just the peaks of all of the humps that is the graph of the cosine function.

Also, when using the continuous function $f(x)$ to evaluate the limit of a sequence $a_n = f(n)$, one can use all of the tools of calculus. But ONLY on the continuous function!

Example 17. Does $a_n = n^2 e^{-3n}$ have a limit? Again, associate to this sequence the continuous function $f(x) = x^2 e^{-3x} = \frac{n^2}{e^{3x}}$. Does $f(x)$ have a horizontal asymptote? Since $f(x)$ is not only continuous but differentiable, we can use all of our calculus tools to study this function. Evaluate $\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}}$. Here, if we push out x to infinity in the expression (in a sense, evaluating the limits of the top and bottom of the fraction), we get the indeterminate form $\frac{\infty}{\infty}$. Thus we can apply L'Hospital's Rule so that

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} e^{3x}} = \lim_{x \rightarrow \infty} \frac{2x}{3e^{3x}}.$$

Repeating the idea, we see again that we would get the indeterminate form $\frac{\infty}{\infty}$. Applying L'Hospital's Rule a second time gets us

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}} = \lim_{x \rightarrow \infty} \frac{\frac{d^2}{dx^2} x^2}{\frac{d^2}{dx^2} e^{3x}} = \lim_{x \rightarrow \infty} \frac{2}{9e^{3x}} = 0.$$

Hence a_n does indeed have a limit.

And now a practical word. One CANNOT use something like L'Hospital's Rule on the sequence a_n . The sequence is NOT differentiable.

More notes:

- if a limit of a sequence exists, we say that the sequence *converges*. if the limit does not exist, then the sequence *diverges*.
- Limits of sequences behave exactly like limits of functions and expression that you learned in an early chapter of the book. You should pay attention to all of the rules on page 678.

Example 18. Product Rule for Limits. Remember the Product rule for Limits: Suppose $\{a_n\}$, and $\{b_n\}$ are two convergent sequences (their limits exist!). Then

$$\lim_{n \rightarrow \infty} a_n b_n = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right).$$

But caution: This ONLY works when both $\{a_n\}$, and $\{b_n\}$ are convergent.

For example, let $\{a_n\} = \{n\}$, and $\{b_n\} = \{\frac{2}{n}\}$. Then

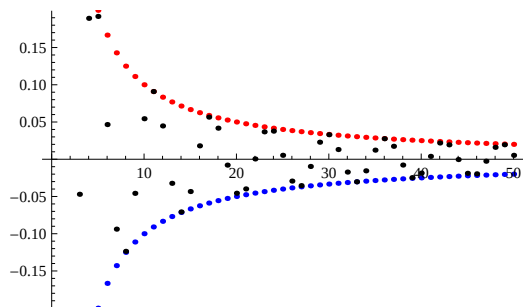
$$2 = \lim_{n \rightarrow \infty} n \cdot \frac{2}{n} \neq \left(\lim_{n \rightarrow \infty} n \right) \left(\lim_{n \rightarrow \infty} \frac{2}{n} \right)$$

since the second limit on the right is 0, while the first is ∞ (it does not exist). Hence the product rule FAILS when one of the individual sequences does not have a limit.

work with these limit rules and get very comfortable with them....

LECTURE 3: SEQUENCES (CONT'D)

Squeezing sequences into limits. It should be quite clear that if one convergent sequence is term-by-term larger than another convergent sequence, then the limit of the first will be bigger than the limit of the second. We can take this one step further by saying that if one sequence lives term-by-term within two other convergent sequences, then the limit of the “sandwiched” sequence, if it exists, will live between the other two. The following theorem follows this theme to a very nice conclusion:



Theorem 19. Suppose $a_n \leq c_n \leq b_n$ for $n > n_0$, and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = L$. Then $\lim_{n \rightarrow \infty} c_n = L$.

Example 20. Show $c_n = \frac{\sin n}{n}$ converges to 0. This should not be hard to see as the numerator never ventures far from 0, while the denominator gets large as the sequence progresses. However, to actually show the result takes a bit of work. One can use the corresponding function $f(x) = \frac{\sin x}{x}$ and try to find an asymptote at infinity, as above. But L'Hospital's Rule doesn't apply here (why not?).

Instead, we will employ the Squeezing Theorem. To start, notice that the numerator satisfies $-1 \leq \sin n \leq 1$ (one of the nice things about this trig function). Divide the inequality by n to get

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Let $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$, and we have

$$a_n \leq c_n \leq b_n$$

just as in the theorem (choose $n_0 = 1$ but any choice of n_0 will do). And since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$, the theorem concludes that also $\lim_{n \rightarrow \infty} c_n = 0$. Hence $\left\{\frac{\sin n}{n}\right\} \rightarrow 0$. See the picture, where a_n is in blue on the bottom, and b_n is in red on top. Both are squeezing c_n in black (the more random-looking points in the middle).

Example 21. Does $c_n = \frac{\ln n}{(n+2)^2}$ converge? This sequence is not as clear as the last example, as both the numerator and the denominator get large as we progress along the sequence. But we can still compare this sequence to others in a straightforward manner.

Notice that for $n > 1$, $c_n > 0$. And notice that, again for $n > 1$, in general

$$\frac{1}{(n+2)^2} < \frac{1}{n^2}.$$

Thus for $n > 1$,

$$0 < c_n = \frac{\ln n}{(n+2)^2} < \frac{\ln n}{n^2}.$$

So let $a_n = 0$ and $b_n = \frac{\ln n}{n^2}$. It is obvious that $\{a_n\} \rightarrow 0$. It is less obvious that $\{b_n\} \rightarrow 0$. I leave this latter calculation to you, but note that the function $f(x) = \ln x x^2$ should be very helpful to you. Then by squeezing, we also know that $\{c_n\} \rightarrow 0$.

Stripping off alternating terms. While many of our examples so far centered on sequence rules that allowed us to use a corresponding function to study their possible convergence, many sequences are not easily converted into functions. For example, sequences with alternating signs, like our original $b_n = \frac{(-1)^n}{2^n}$ do not work so well as continuous functions by replacing nc with x . However, there is hope, and it involves stripping away the alternating term.

Theorem 22. *If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.*

One must be careful here to apply this theorem correctly. It ONLY works when the limit of the absolute values of the sequence is 0, and not another number. The quintessential example of this is the sequence $a_n = (-1)^n$. Here

$$\lim_{n \rightarrow \infty} |(-1)^n| = \lim_{n \rightarrow \infty} 1 = 1, \text{ but } \lim_{n \rightarrow \infty} (-1)^n \text{ does not exist!}$$

Example 23. *Show $b_n = \frac{(-1)^n}{2^n}$ converges to 0.* We use the theorem directly, since

$$\lim_{n \rightarrow \infty} |b_n| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{2^n} \right| = \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0 \text{ (Why?)}$$

hence by the theorem, so does b_n .

Passing a limit through a continuous function. One of the endearing properties of limits is that one can pass a limit through a continuous function. Essentially this is Theorem 8 in Section 2.5 on continuity in the text (see page 125). For sequences, this remains true:

Theorem 24. *If $\lim_{n \rightarrow \infty} a_n = L$ and f is a function continuous at L , then*

$$\lim_{n \rightarrow \infty} f(a_n) = f(L).$$

This is particularly useful when the sequence rule is quite complicated and consists of a composition of functions.

Example 25. *Show $a_n = \tan\left(\frac{\pi n^2}{9n+4n^2-12}\right)$ converges and find the limit.* We will use this theorem to solve this problem by first looking at the sequence $b_n = \frac{\pi n^2}{9n+4n^2-12}$. We find that this sequence is convergent and

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\pi n^2}{4n^2 + 9n - 12} = \frac{\pi}{4} \text{ (again, why?).}$$

And since $f(x) = \tan x$ is continuous at $x = \frac{\pi}{4}$, we have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \tan(b_n) = \tan\left(\lim_{n \rightarrow \infty} b_n\right) = \tan \frac{\pi}{4} = 1.$$

To go farther, let's look more closely at some of the properties of certain sequences.

Definition 26. A sequence $\{a_n\}$ is called *increasing* if, for all $n \geq 1$, we have $a_n < a_{n+1}$. $\{a_n\}$ is called *decreasing* if, for all $n \geq 1$, we have $a_n > a_{n+1}$. If either is the case, the sequence is called *monotonic*.

Definition 27. A sequence $\{a_n\}$ is said to be *bounded above* if there is a constant M , where $a_n \leq M$ for all $n \geq 1$. $\{a_n\}$ is said to be *bounded below* if there is a constant m , where $a_n \geq m$ for all $n \geq 1$. If both are true, the sequence is simply called *bounded*.

This immediately leads to yet another theorem on the convergence of sequences:

Theorem 28. *Every bounded monotonic sequence converges.*

Really, think of this as follows: If a sequence must always rise, and yet never cross a particular value, then it must converge. Think of a helium balloon in a closed room. It cannot go through the ceiling, and yet will constantly rise (at least until the helium leaks out, that is). Hence at least in the limit, it will have to reach a steady height.

Sequences can also be defined recursively, as in the value of a particular term is a function, not of the index, but of the value of the previous term or some combination of previous terms of the sequence. For example, Example 13 in the section defines the sequence

$$a_{n+1} = \frac{1}{2}(a_n + 6), \quad a_1 = 2.$$

And Example 3 in the book defines the Fibonacci Sequence as

$$f_{n+2} = f_{n+1} + f_n, \quad f_0 = 1, f_1 = 1.$$

Notice that the rule in both of these cases specifies the value of a term as a function of the previous terms. Notice also that for this to work, we would always need to “start” somewhere. The obvious advantage to this is that for different starting values, we can get many different sequences, whose properties like convergence may depend on where one starts. The obvious disadvantage to this is that to know the value of a term whose index is large would require knowing ALL of the previous terms’ values. That could be problematic. Compare that to the functional form of most of our sequences that we have defined so far.

Pay attention to Example 13 in the text. There are great conceptual ideas in this discussion. One such idea involves the notion of *mathematical induction*: a form of rigorous deductive reasoning that states that a statement is deemed true for all of the natural numbers if it is true for the smallest of them, and if it is also true at any stage, then it is true also at the next stage. It is a great way to establish a general result with only a couple of calculations, and forms one of the standard ways that mathematicians prove things.

In the case at hand, let’s study the sequence of Example 13. Notice first what the sequence will be if we start with $a_1 = 6$. We get

$$a_{n+1} = \frac{1}{2}(a_n + 6), \quad a_1 = 6 \implies a_n = 6, \text{ for all } n \in \mathbb{N}.$$

Now suppose that we start with some positive starting value for the sequence $0 \leq a_1 < 6$. Then, it is the case that

$$0 \leq a_2 = \frac{1}{2}(a_1 + 6) < \frac{1}{2}(6 + 6) = 6.$$

In fact, let’s suppose that somewhere along our sequence the n th term satisfies $0 \leq a_n < 6$. Then $6 \leq a_n + 6 < 6 + 6 = 12$. Then also $\frac{1}{2}(6) = 3 \leq \frac{1}{2}(a_n + 6) < \frac{1}{2}(12) = 6$. But the middle part of the inequality IS a_{n+1} . Hence we have shown that whenever a term in the sequence is between 0 and 6, then so is its successor. Hence if we start in the interval $[0, 6]$, we never leave it. By induction, the sequence $\{a_n\}$ is **bounded**.

Let’s play this game again in a different way. Notice that for our original sequence starting at $a_1 = 2$, we get $a_2 = \frac{1}{2}(2 + 6) = 4$, which is greater than 2. we can duplicate this again through out the sequence to see how the sequence progresses.

Suppose we knew that somewhere along the sequence we had $a_{n-1} < a_n$. Then we can say

$$\begin{aligned} a_{n-1} &< a_n \\ a_{n-1} + 6 &< a_n + 6 \\ \frac{1}{2}(a_{n-1} + 6) &< \frac{1}{2}(a_n + 6) \\ a_n &< a_{n+1}. \end{aligned}$$

Thus if somewhere along the way we know the sequence rises, then at the next stage the sequence continues to rise. Since it started by rising, it will be forced to rise forever (but never get larger than 6, right?). Hence, by induction, the sequence is increasing, and **monotonic**.

Put these together and you get by the previous theorem that this sequence must converge. It turns out that it converges to 6, actually.