110.109 CALCULUS II

Week 6 Lecture Notes: March 7 - March 11

Arc-length in Polar Coordinates

Today is a short day, due to the fact that for the second half of this lecture, I want to focus on the midterm exam coming up next class, its format and the areas you will be responsible for.

However, for today, I want to talk about the ability, given a polar curve in the form of a function $r = f(\theta)$, to calculate the length of the curve, given a range of values for the dependent variable θ.

In this regard, let $r = f(\theta)$ be a polar curve. We know, given a function $y = F(x)$, that the length of a curve, from $x = a$ to $x = b$, is given by the formula

$$\text{Length } = \int_a^b \sqrt{1 + \left(F'(x)\right)^2}\, dx.$$

To review this, go to Section 8.1. This is part of the syllabus for the course 110.108 Calculus I. we also know that, given a parameterization of the same curve $x(t)$ and $y(t)$, that the curve is $y(t) = F\left(x(t)\right)$, that

$$\text{Length } = \int_a^b \sqrt{1 + \left(F'(x)\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt}\, dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

where $x(\alpha) = a$ and $x(\beta) = b$. Note that this is really just a substitution type argument, and is what I call the Anti-Chain Rule. See last week's lectures for details, or Section 10.2.

We seek to play the same game for a polar curve. Indeed, for any polar curve $r = f(\theta)$, we can re-write the equations that relate polar coordinates back to rectilinear coordinates

$$x = r\cos\theta = f(\theta)\cos\theta) \text{ and } y = r\sin\theta = f(\theta)\sin\theta).$$

In this way, curve is written in the rectilinear coordinates and parameterized by θ. Then the formula for arc length comes directly from Section 10.2:

$$\text{Arc Length } = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This is workable, but having to go back to rectilinear coordinates to calculate the length of a polar curve is not optimal. Better to be able to calculate directly, no? It turns out that we can. First, write out the derivatives of the parameterization:

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{d}{d\theta}\left(r\cos\theta\right) = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ \frac{dy}{d\theta} &= \frac{d}{d\theta}\left(r\sin\theta\right) = \frac{dr}{d\theta}\sin\theta + r\cos\theta. \end{aligned}$$

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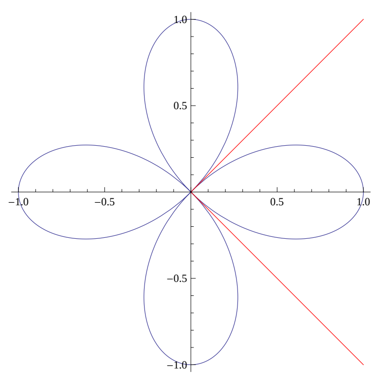
Note that we need the product rule here because r is a function of θ. Back to the formula for arc length, we place this in and look for simplifications. We get

$$\begin{aligned}
\text{Arc Length} &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\cos^2\theta}\, d\theta \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\sin^2\theta + \cos^2\theta\right)}\, d\theta \\
&= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta
\end{aligned}$$

We now have a formula to calculate the length of a polar curve directly.



Example 1. *Set up the calculation to find the length of the perimeter of of one leaf of a 4-leaf rose $r = \cos 2\theta$.* Here, the 4-leaf rose has one of its petals lying symmetrically on the polar axis (the positive x axis in rectilinear coordinates). The integrand of the arc-length integral is

$$\begin{aligned}
\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} &= \sqrt{(-2\sin 2\theta)^2 + (\cos 2\theta)^2} \\
&= \sqrt{4\sin^2 2\theta + \cos^2 2\theta} = \sqrt{3\sin^2 2\theta + 1},
\end{aligned}$$

since $\frac{dr}{d\theta} = -2\sin 2\theta$, and we use the standard identity $\sin^2 x + \cos^2 x = 1$ to help simplify what is under the radical.

We still need to know where to integrate, though (where the limits are). Essentially, we need to find the interval of θ where the leaf is traced exactly once. In this case, it is fairly easy: Look for two consecutive places where $r = 0$. We can solve:

$$\begin{aligned}
r = 0 = \cos 2\theta \iff 2\theta &= \frac{\pi}{2} \iff \theta = \frac{\pi}{4}, \text{ and} \\
\iff 2\theta &= -\frac{\pi}{2} \iff \theta = -\frac{\pi}{4}.
\end{aligned}$$

These are the two red lines in the figure. Our calculation is then

$$\text{Arc Length} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{3\sin^2 2\theta + 1}\, d\theta$$

and we are done.

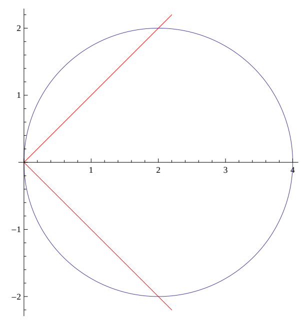
Example 2. *Find the length of the graph of the one-leaf rose $r = 4\cos\theta$.* Here, $f(\theta) = 4\cos\theta$, and $f'(\theta) = -4\sin\theta$. What are the limits of the arc-length integral? The interval of θ where the leaf is traced out once. Recall that the off-leaf roses are traced twice over the full range of θ. Here, any interval of length

π will do. We choose $\theta = 0$ and $\theta = \pi$ for now. Our length then is

$$\text{Arc Length } = \int_0^\pi \sqrt{(4\cos\theta)^2 + (-4\sin\theta)^2}\, d\theta = \int_0^\pi \sqrt{16\cos^2\theta + 16\sin^2\theta}\, d\theta = \int_0^\pi \sqrt{16}\, d\theta = 4\theta\Big|_0^\pi = 4\pi.$$

So what is the perimeter of a circle of radius 2 (see figure below)?

And lastly, we talked about tangents to polar curves. I neglected to discuss this when I was doing Section 10.3, because it is very straightforward, and wanted to spend more time on other stuff. But I will talk about it here.



Firstly, given any polar curve $r = f(\theta)$, notice that the equations converting polar coordinates to rectilinear coordinates $x = r\cos\theta$ and $y = r\sin\theta$, are actually simply funcitons of θ when restricted to the polar curve:

$$x(\theta) = f(\theta)\cos\theta \text{ and } y(\theta) = f(\theta)\sin\theta.$$

Thus the polar curve is just a parameterized curve where θ is the parameter. So the calculation of the slope of the curve is basically the same as that of Section 10.2:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

While this looks much harder to calculate in general due to the trigonometric functions, the theory involves nothing more than this expression.

Example 3. *Find the values of θ where the slope of the line tangent to $r = 4\cos\theta$ is horizontal and vertical.* Here $\frac{dr}{d\theta} - r\sin\theta$, so the slope of the tangent line at a value of θ is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-4\sin\theta\sin\theta + 4\cos\theta\cos\theta}{-4\sin\theta\cos\theta - 4\cos\theta\sin\theta}.$$

Finding the horizontal lines means setting this last expression to 0. Hence

$$\begin{aligned} -4\sin\theta\sin\theta + 4\cos\theta\cos\theta &= 0 \\ \sin^2\theta &= \cos^2\theta, \end{aligned}$$

which is solved precisely when either $\sin\theta = \cos\theta$, or $\sin\theta = -\cos\theta$. It turns out this happens rather often, when $\theta = (2n+1)\frac{\pi}{4}$, $n \in \mathbb{Z}$. This is any ODD integer multiple of $\frac{\pi}{4}$. But given the graph above, ANY odd multiple of $\frac{\pi}{4}$ lands on one of the two points at the top and bottom of the circle given by θ_1 and θ_2. See the graph. The two red lines correspond to $\theta = \pm\frac{\pi}{4}$.

How would one go about finding the vertical tangents? Really, just set the denominator to 0 and solve. You should get $\theta = n\frac{\pi}{2}$, $n \in \mathbb{Z}$: The boiled-down equation is $\sin\theta\cos\theta = 0$.

Lecture 2: Midterm 1 day, no lecture

Lecture 3

Today we started Section 7.8, on Improper Integrals. To start the discussion, consider the integral

$$\int_1^2 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

This is simply a straightforward application of the Anti-Power Rule, and since the function is positive for all $x > 0$, the definite integral represents the area between the curve $f(x) = \frac{1}{x^2}$ and the x-axis. if we changed the upper limit from 2 to a 5, or to a 10, or to a 1000, how would the calculation change? Indeed, for ANY $b > 0$ (including $0 < b < 1$, we get

$$\int_1^b \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^b = -\frac{1}{b} + 1 = \frac{b-1}{b}.$$

Looking at this form, answer the following question:

Question. *Is there a $b > 1$ where $\int_1^b \frac{1}{x^2}\,dx > 1$?*

The answer, by looking at the form $\frac{b-1}{b}$, is most certainly not. Then can we say something about whether the following limit exists and what it is (if it exists):

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx.$$

Indeed, we can. Since for any $b > 1$, the integral is a definite integral of a positive function, the value of the definite integral, as the area of the region bounded by the function and the x-axis on the interval $[1, b]$ is positive. And since the integral can be expressed simply as a expression involving only b, we can then evaluate the limit using techniques from back in Calculus I:

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{x}\bigg|_1^b \right) = \lim_{b\to\infty} \left(-\frac{1}{b} + 1 \right) = \lim_{b\to\infty} \frac{b-1}{b} = 1.$$

This means the the area of the *unbounded* region represented between the function and the x-axis on the infinite interval $[1, \infty)$, is bounded and equal to 1. Really what we are asking for is the quantity

$$\int_1^\infty \frac{1}{x^2}\,dx,$$

but the limit does not make sense given our rules for integration. It is for this reason that we call such an integral an *improper integral*, and define it as such:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx.$$

Definition 4. For $f(x)$ continuous for all $x \geq a$, the improper integral

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

provided the limit exists. If the limit exists, we say the improper integral *converges*. Else, we say it *diverges*.

The effect of this calculation is that one calculates the definite integral for some b as the upper limit, giving an area (if the function is positive. Else the value of the integral loses it's interpretation as an area.) in terms of b. Then we push b to infinity and watch how the value changes. If the value-changes settle down to a number (in the limit), then that number IS the value of the improper integral. There are many examples, some of which I gave in class, and many more in the book. Here is one.

Example 5. Calculate $\int_1^\infty \frac{1}{x}\,dx$, if it exists. Here

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} (\ln|x|)\bigg|_1^b = \lim_{b\to\infty} (\ln|b| - \ln|1|) = \lim_{b\to\infty} \ln b = \infty,$$

hence does not exist (the improper integral diverges).

There are further details to the definition above:

Definition 6. For $f(x)$ continuous for all $x \leq b$, the improper integral

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx,$$

provided the limit exists.

Example 7. Calculate $\displaystyle\int_0^\infty te^{-t}\,dt$, if it exists (Note that this is much like Example 7.8.2 in the book, but going the other way). Here the function is continuous everywhere, but the integral is improper. Hence we need to appeal to the limit:

$$\int_0^\infty te^{-t}\,dt = \lim_{b\to\infty} \int_0^b te^{-t}\,dt.$$

Using the technique of Integration by Parts, where $f(t) = t$, and $g'(t) = e^{-t}$, we get $f'(t) = 1$, and $g(t) = -e^{-t}$, and

$$\begin{aligned}
\lim_{b\to\infty} \int_0^b te^{-t}\,dt &= \lim_{b\to\infty} \left(-te^{-t}\Big|_0^b - \int_0^b -e^{-t}\,dt \right) \\
&= \lim_{b\to\infty} \left(-te^{-t} - e^{-t}\,dt\right)\Big|_0^b \\
&= \lim_{b\to\infty} \left(\left(-be^{-b} - e^{-b}\right) - (0-1)\right) \\
&= \lim_{b\to\infty} \left(-\frac{b+1}{e^b} + 1 \right) = 1.
\end{aligned}$$

How would you show the last step? Try L'Hospital's Rule. Hence, this integral converges and its value is 1.

Example 8. Calculate $\displaystyle\int_{-\infty}^{9} \sin x\,dx$, if it exists. Here the function is continuous everywhere, and the integral is again improper. Appealing to the limit:

$$\int_{-\infty}^{0} \sin x\,dx = \lim_{a\to-\infty} \int_a^0 \sin x\,dx = \lim_{a\to-\infty} (-\cos x)\Big|_a^0 = \lim_{a\to-\infty} (-\cos 0 + \cos a)\,.$$

But this limit does not exist for a different reason; the cosine function does nto have a limit at infinity (it does not have a horizontal asymptote!). Hence the improper integral diverges.

And finally, an integral is also improper if both of its limits "run off the page". Well, to handle this case, remember that we can always break up an interval given by the limits into two pieces, evaluate the definite integral on both pieces, and then add the results. This is important for the following definition.

Definition 9. Suppose $f(x)$ is continuous on $\mathbb{R}$. Then for ANY choice of $c \in \mathbb{R}$, the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

provided that both of the improper integrals on the right-hand-side exist.

Example 10. Calculate $\displaystyle\int_{-\infty}^{\infty} 3xe^{-x^2}\,dx$, if is exists. Here, the integrand is continuous on the entire real line (so the integral will exist on ANY finite interval), and the integral is improper (both limits need to be addressed). Choose our intermediate value to be 0 for symmetry (though this does not matter), and

$$\begin{aligned}\int_{-\infty}^{\infty} 3xe^{-x^2}\,dx &= \int_{-\infty}^{0} 3xe^{-x^2}\,dx + \int_{0}^{\infty} 3xe^{-x^2}\,dx \\ &= \lim_{a\to-\infty}\int_{a}^{0} 3xe^{-x^2}\,dx + \lim_{b\to\infty}\int_{0}^{b} 3xe^{-x^2}\,dx.\end{aligned}$$

Here, a straightforward substitution of $u = x^2$, $du = 2x\,dx$ is very helpful. Working just with the antiderivative for a minute, we get

$$\int 3xe^{-x^2}\,dx = \int \frac{3}{2}e^{-u}\,du = -\frac{3}{2}e^{-u} + C = -\frac{3}{2}e^{-x^2} + C.$$

Back to our improper integral, we have

$$\begin{aligned}\int_{-\infty}^{\infty} 3xe^{-x^2}\,dx &= \lim_{a\to-\infty}\int_{a}^{0} 3xe^{-x^2}\,dx + \lim_{b\to\infty}\int_{0}^{b} 3xe^{-x^2}\,dx \\ &= \lim_{a\to-\infty}\left(\left(-\frac{3}{2}e^{-x^2}\right)\Bigg|_{a}^{0}\right) + \lim_{b\to\infty}\left(\left(-\frac{3}{2}e^{-x^2}\right)\Bigg|_{0}^{b}\right) \\ &= \lim_{a\to-\infty}\left(-\frac{3}{2} + \frac{3}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{3}{2}e^{-b^2} + \frac{3}{2}\right) = \frac{3}{2} + \frac{3}{2} = 3.\end{aligned}$$

The improper integral converges to 3.

Lastly, I defined another way for an integral to be improper. I will say more about this on Monday.