110.109 CALCULUS II

Week 11 Lecture Notes: April 18 - April 22

Lecture 1: Section 10.9 Representing Functions as Power Series

Let's go back to our geometric series, written as a function where the series converges:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \text{ for } -1 < x < 1.$$

This idea is fairly profound, as it equates a function like the rational function on the right with an infinite degree polynomial. Keep in mind, though, that the domain of the function $\frac{1}{1-x}$ is all $x \neq 1$, and the series only converges for $x \in (-1, 1)$. Hence, this series equals the function only on the domain where the series actually makes sense (read: converges).

Other functions that can be manipulated into looking like the one above can now also be written as power series. For some examples:

Example 1. Let $g(x) = \frac{1}{1-x^2}$. Here we can see directly that $g(x) = f(x^2)$, and $g(x)$ is a composite function of the above $f(x)$ with x^2. Hence we can write

$$g(x) = \frac{1}{1-x^2} = \frac{1}{1-(x^2)} = f(x^2) = \sum_{n=0}^{\infty} \left(x^2\right)^n = \sum_{n=0}^{\infty} x^{2n}.$$

Again, we would need to know where this series converges to know the domain for which we can equate the function $g(x)$ to the series. By the Root Test, where $a_n = x^{2n}$, we get that the series will converge where

$$\lim_{n\to\infty} \sqrt[n]{|x^{2n}|} = \lim_{n\to\infty} \sqrt[n]{\left(x^2\right)^n} = \lim_{n\to\infty} x^2 = x^2 \lim_{n\to\infty} 1 = x^2 < 1.$$

Hence the radius of convergence is $R = 1$. As for the endpoints, when $x = 1$ and when $x = -1$, the series does not converge (can you see this? Use the Divergence Test on these two series. You will find the terms do not go to 0). Hence we have that the interval of convergence is $(-1, 1)$, and

$$g(x) = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \ldots \text{ for } x \in (-1, 1).$$

Example 2. Let $h(x) = \frac{1}{1+x^2}$. Here again we can see that $h(x) = f(-x^2)$, and $h(x)$ is a composite function of the above $f(x)$ with $-x^2$. Hence we can write

$$h(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = f(-x^2) = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

As for where this alternating series converges, we appeal to the Ratio Test, where $a_n = (-1)^n x^{2n}$ and $a_{n+1} = (-1)^{n+1} x^{2(n+1)}$, we get that the series will converge where

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} x^2 = x^2 \lim_{n\to\infty} 1 = x^2 < 1.$$

Again, the radius of convergence is $R = 1$, and again when $x = 1$ and when $x = -1$, the series does not converge (you should work this out explicitly?). Hence the interval of convergence is $(-1, 1)$, and

$$h(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots \text{ for } x \in (-1, 1).$$

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Example 3. Let $i(x) = \frac{1}{3-x}$. This time, using a bit of algebraic manipulation, we can write

$$i(x) = \frac{1}{3-x} = \frac{1}{1-(x-2)} = f(x-2),$$

and $i(x)$ is a composite function of the above $f(x)$ with $x-2$. Hence

$$i(x) = \frac{1}{1-(x-2)} = \sum_{n=0}^{\infty} (x-2)^n.$$

This is the standard geometric series centered on 2, instead of 0. As for where this alternating series converges, really, all of these are geometric, so one can almost guess where each of these series in these examples will converge. This one again will have a radius of convergence of $R = 1$, and the endpoints are not included. Careful here as the final result is that the interval of convergence is centered at 2, and

$$i(x) = \frac{1}{3-x} = \sum_{n=0}^{\infty}(x-2)^n = 1 + (x-2) + (x-2)^2 + (x-2)^3 + \dots \text{ for } x \in (1,3).$$

Example 4. Let $j(x) = \frac{1}{3+x}$. This time, we need a bit more work, algebraically. Here

$$j(x) = \frac{1}{3+x} = \frac{1}{3(1+\frac{x}{3}} = \frac{1}{3}\left(\frac{1}{1-\left(-\frac{x}{3}\right)}\right) = \frac{1}{3}f\left(-\frac{x}{3}\right).$$

For the series, then

$$j(x) = \frac{1}{3}\left(\frac{1}{1-\left(-\frac{x}{3}\right)}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{3^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{3^{n+1}}.$$

The last little manipulation is not really necessary, but does clean things up a bit. Again, for the radius of convergence, we use the Ratio Test with $a_n = (-1)^n\frac{x^n}{3^{n+1}}$, and $a_{n+1} = (-1)^{n+1}\frac{x^{n+1}}{3^{n+2}}$ and get

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{x^{n+1}}{3^{n+2}}}{(-1)^n\frac{x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|\lim_{n\to\infty}1 = \left|\frac{x}{3}\right| < 1,$$

which is satisfies for $|x| < 3$. With the extra knowledge that the endpoints are not included (check this again), we have that the interval of convergence is $(-3,3)$ and the final result is

$$j(x) = \frac{1}{3+x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{3^{n+1}} = \frac{1}{3} - \frac{x}{9} + \frac{x^2}{27} + \frac{x^3}{81} + \dots \text{ for } x \in (-3,3).$$

Notice that infinite power series act just like infinite degree polynomials. hence one can differentiate and integrate them term by term (the derivative/integral of a sum is the sum of the derivatives/integrals, no?) using nothing more than the Power/Anti-Power Rules:

Theorem 5. *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

be continuous and differentiable on the interval $(a-R, a+R)$. *Then*

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

and

$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

The radius of convergence for each of these is equal to the original radius of convergence.

Remark 6. One note, here: Notice that for the anti-derivative, we needed to introduce the constant of integration C, as usual. Notice also that the anti-derivative of the constant term c_0 is $c_0(x-a)$ instead of simply c_0x. This is done purely for symmetry. It seems like we added a new term $-ac_0$ to the mix. But with the constant of integration being an unknown, this extra term is just another constant and gets absorbed once one finds the value of C in a real problem. Plus, it makes the calculation as a series very efficiently specified, so this is why it is included this way.

Example 7. *Express* $g(x) = \frac{1}{(1-x)^2}$ *as a power series.* Here, this function does not look fit to be expressed as a geometric power series directly. However, with a little foresight, we can see that for $f(x) = \frac{1}{1-x}$, that

$$f'(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{-1}{(1-x)^2)}(-1) = \frac{1}{(1-x)^2} = g(x).$$

This is very helpful, given the previous theorem, since $f(x)$ can be written as our standard power series, and we can differentiate it to get $g(x)$. So

$$g(x) = f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n.$$

Again, the last manipulation is not really needed. And again, the Radius of convergence does not change here, so $g(x)$ equals this power series on the interval $(-1,1)$.

Here are a couple of notes:

- The series corresponding to $g(x)$ above is NOT geometric, given the coefficients are a non-constant function of n. It was derived (literally) from a geometric series, though. Geometric series are very special, and somewhat rare.
- while the radius of convergence does NOT change under differentiation and integration, the interval of convergence may change. The endpoints will always need to be checked after differentiation or integration. Sorry about that.

Example 8. *Write* $\ell(x) = \ln\left(1+x^2\right)$ *as a power series.* Here, this function does not look remotely like a geometric series. However, it is related in a way. First, notice that

$$\ell'(x) = \frac{d}{dx}\left[\ln\left(1+x^2\right)\right] = \frac{2x}{1+x^2} = 2x\left(\frac{1}{1-(-x^2)}\right).$$

Written like this last term on the right, one can see that the derivative does indeed look like the term $2x$ multiplied by the function in Example 2 above. We already know how to write that function as a power series, and thus

$$\ell'(x) = 2x\left(\frac{1}{1-(-x^2)}\right) = 2x\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right) = \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1}.$$

We can check immediately that the interval of convergence here is again $(-1,1)$ (which you should do).

This gives us the series representation for $\ell'(x)$. We want $\ell(x)$, which we can find via integration:

$$\ell(x) = \ln\left(1+x^2\right) = \int\left(\sum_{n=0}^{\infty} 2(-1)^n x^{2n+1}\right)dx = C + \sum_{n=0}^{\infty} 2(-1)^n\frac{x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1},$$

where we found a way to kill off the 2 in the series terms (not necessary, actually). We need to find the correct value of C here, and since $\ell(0) = \ln\left(1 + 0^2\right) = 0$, we find that $C = 0$, and

$$\ln\left(1 + x^2\right) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \ldots \text{ for } x \in (-1, 1).$$

Note that Example 7, page 731 in the book is another good example to study. We end with one more:

Example 9. *Express $f(x) = \frac{1+x}{1-x}$ as a power series.* Very quickly, notice that

$$f(x) = \frac{1+x}{1-x} = \frac{1}{1-x} + \frac{x}{1-x} = (1+x)\frac{1}{1-x}.$$

Hence we can write

$$f(x) = (1+x)\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty}(1+x)x^n = \sum_{n=1}^{\infty} x^n + x^{n+1}.$$

Write this in any way you choose. Calculate the interval of convergence here (you will find $R = 1$). And on this interval we have

$$\begin{aligned}
\frac{1+x}{1-x} &= (1+x) + (1+x)x + (1+x)x^2 + (1+x)x^3 + \ldots \\
&= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 \ldots \\
&= (-1) + (2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \ldots) \\
&= (-1) + 2(1 + x + x^2 + x^3 + x^4 + x^5 + \ldots) \\
&= (-1) + 2\sum_{n=1}^{\infty} x^n,
\end{aligned}$$

although seeing this last part is again not necessary at all.