110.109 CALCULUS II

Week 3 Lecture Notes: February 14 - February 18

LECTURE 1

Today, I started with some basics about differential equations (called ordinary differential equations, or ODEs). I began with the definition of the order of an ODE, defined as the number corresponding to the highest derivative found in the equation. Some examples were: $P'(t) = kP(t)$, a first order ODE and an example of the exponential growth found in population dynamics, $\ddot{x} + k\dot{x} + \sin x = 0$, a second order ODE and a model for the pendulum, and $\left[y_{(5)}(t)\right]^2 - e^x y^{(3)}(t) + (\sin t)y'(t) = e^t$, a fifth-order nonsense ODE (Note the notation when talking about higher order derivatives). I talked about the general components of a first order ODE; the independent variable, usually called time, the dependent variable, the unknown function of "time", and the derivative of the unknown function. And I talked about a relatively general form for many first order ODEs. That is

$$y' = f(t, y). \tag{1}$$

The right hand side of this last expression is considered some expression involving both the dependent cariable y and the independent variable t, and can be almost anything function-like. Sometimes the expression on the right hand side does not include the y variable. Then $y'(t) = f(t)$. if this is the case, then the ODE is called a "pure-time" ODE, and solving is simply integrating. The solution (solving for the unknown function) is simply the antiderivative of $f(t)$, and $y(t) = \int f(t)\,dt = F(t) + C$, where $F'(t) = f(t)$. As an example, I did the ODE $y' = 2x + xe^x$. The solution is

$$y(x) = \int (2x + xe^x)\,dx = x^2 + (x-1)e^x + C.$$

Note when the right hand side of Equation 1 involves both y and t, solving the ODE is more tricky, and we will study dome of these ways in this chapter. As an example, I asked you to show that $y(t) = \frac{t^4}{t^6+C}$ solve the ODE $y' = \frac{4y}{t} - 6ty^2$. Really, this is done simply by taking the derivative of the solution, and the original solution and subbing appropriately into the original ODE. I then talked about the initial value and gave three examples of such. Like integrating in general, it turns out that there are tone of solutions to every ODE, but they all differ by a constant (like antiderivatives). If you solved an ODE, and also knew one bit of information about it, you can use that to find a value for the unknown constant of integration you have in your general solution. For example, solve $y' = \frac{4y}{t} - 6ty^2$ with the additional information (the initial value) that $y(1) = 2$. From above, we knew that the solution functions to the ODE all look like $y(t) = \frac{t^4}{t^6+C}$. The one we want needs to fit the bit of data we have. We solve for it:

$$y(1) = \left.\frac{t^4}{t^6 + C}\right|_{t=1} = \frac{1}{1+C} = 2.$$

We see that $C = -\frac{1}{2}$, and the solution we want is $y(t) = \frac{t^4}{t^6 - \frac{1}{2}}$.. I then launched into a discussion of a visual analysis of a first order ODE called a slope field. I defined it as a grid of small line segments of the xy-plane, each of which is tangent to a graph of a solution $y(x)$ of the ODE $y' = f(x, y)$. Finding these small line segments is easy. The slope of each of the line segments is given by the function $f(x, y)$. This is because for ANY solution $y(x)$, its derivative at any value x is precisely $y'(x) = f(x, y)$. The ODE itself gives you all of this information. The beauty of the slope field is that it acts like grasses at the bottom of a swiftly moving stream; The current of the water is displayed by how each blade of grass is bent over. Drop a leaf into the stream, and the path of the leaf will follow the "flow" lines of the grasses. It will follow the "solution" given by the slope field. I tried to show this electronically to the class via some software, but the computer in Krieger 205 has been compromised by Spyware/Adware. Hence on Wednesday, I will bring in my laptop.

Date: February 15, 2011.