MATH 421 DYNAMICS

Week 8 Lecture 2 Notes

---

1. Newtonian Systems of Classical Mechanics (cont'd.)

Last class we began a discussion on a particular type of mathematical models; those given by a second-order differential equation. We will see that they exhibit the phase space incompressibility we mentioned. Their structure actually exposes a lot about the dynamical behavior of systems of this type. First, some easy examples:

Example 1. An object under the influence of gravity:

$$\ddot{x} = -g, \Longrightarrow s(t) = -g\frac{t^2}{2} + v_0 t + s_0,$$

where v_0 and s_0 are the initial velocity and initial position, respectively.

Example 2. Harmonic Oscillator. Recall Hooke's Law: the amount an object is deformed is linearly related to the force causing the deformation. This translates to

$$\ddot{x} = -kx, \Longrightarrow x(t) = a\sin\sqrt{k}t + b\cos\sqrt{k}t,$$

where a and b are related to the initial starting position and velocity of the mass.

As stated above, note that any equation ODE of the form $\ddot{x} = f(t, x, \dot{x})$ can be converted to a system of two first order (generally coupled) ODEs of the form

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= f(t, x, v) \end{aligned}$$

which defines a vector field (static one if t does not appear explicitly in the equations) on the (x, v)-state space. For Newton's Equation, the equivalent system is $\dot{x} = v$ and $\dot{v} = \frac{1}{m}f(x)$.

Note: For the n-system governed by Newton's Law $f = ma$, we get the $2n$-system of first order equations defined as

$$\begin{aligned} \dot{\overline{x}} &= \overline{v} \\ \dot{\overline{v}} &= \overline{f(t, \overline{x}, \overline{v})} \end{aligned}$$

The state space consists of the $2n$-dimensional vectors $\begin{bmatrix} \overline{x} \\ \overline{v} \end{bmatrix}$. The vector field of this $2n$-system attaches the vector $V = \begin{bmatrix} \overline{v} \\ \frac{1}{m}\overline{f(\overline{x})} \end{bmatrix}$ to each point $\begin{bmatrix} \overline{x} \\ \overline{v} \end{bmatrix}$. The divergence of this vector field is

$$\nabla \cdot V = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(v_i) + \sum_{i=1}^{n} \frac{\partial}{\partial v_i}\left(\frac{1}{m} f_i(\overline{x})\right) = 0.$$

Hence the flow preserves volume.

This fact is now true for general Newtonian systems. One facilitating idea in Newtonian physics is to, in essence, factor out the mass. Define a new variable $p := mv$ as the (linear) momentum. Then $f = ma = f = m\dot{v} = \dot{p}$, and the system becomes $\dot{x} = \frac{p}{m}$ a,d $\dot{p} = f(x)$. Not only does this make the system easier to work with, it exposes some hidden symmetries within the equations of conservative systems.

Date: March 30, 2010.

Now assume that the force $f(x)$ is a gradient field (this means that the force is the gradient of a function of position alone, or $f = -\nabla V$ for some $V(x)$. Then

$$f(x) = ma = m\dot{v} = -\nabla V.$$

We could also say that

$$\begin{aligned} \dot{x} &= \frac{p}{m} \\ \dot{p} &= -\nabla V(x). \end{aligned}$$

Here, the function V is called the potential energy (energy of position), and the energy of motion, the kinetic energy is

$$K = \frac{1}{2}m\,||v||^2 = \frac{1}{2}m(v\dot{v}).$$

The total energy $H = K + V$ satisfies

$$\frac{d}{dt}(H) = \frac{dK}{dt} + \frac{dV}{dt} = m\dot{v}\cdot v + \sum_{i=1} n\frac{\partial V}{\partial x_i}\cdot\frac{\partial x_i}{\partial t} = m\dot{v}\cdot v + \nabla V \cdot v = (\nabla V + m\dot{v}, v) = 0.$$

The conclusion is the total energy H is conserved as one evolves in a system like this. As H is a function defined on the state space given by the vectors x and mv, the solutions to the system of ODEs are confined to the level sets of this function. A system like this is called *conservative*, and is characterized by the idea that the force field is a gradient field. You have seen this before in a different guise:

Example 3. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= 4 - 2y \\ \dot{y} &= 12 - 3x^2. \end{aligned}$$

This system can also be written by the single differential equation $(12 - 3x^2)dx - (4 - 2y)dy = 0$. Note that this equation is exact, and separable, and upon integration, one obtains

$$4y - y^2 = 12x - x^3 + C.$$

this defines our solutions implicitly. In fact, we can use this directly. Define a function

$$H(x, y) = 4y - y^2 - 12x + x^3.$$

Then H is conserved by the flow, and the flow must live along the constant level sets of H (the sets that satisfy $H(x, y) = C$.) These sets are given by the figure.