MATH 421 DYNAMICS

Week 8 Lecture 1 Notes

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1. Incompressibility (cont'd.)

We ended last class defining the notion of phase volume preservation, or incompressibility of a flow on the phase space. We ended with a statement that if the flow has all time-t maps are isometries (or for a discrete dynamical system, if a map is an isometry), then the distances between each pair of points remains constant under the transformation(s), and the flow (map) is incompressible. However, a transformation being an isometry is not a necessary condition for incompressibility.

Example 1. Consider the linear twist map on the cylinder

$$T : S^1 \times [0, 1] \longrightarrow S^1 \times [0, 1], \quad T(x, y) = (x + y, y).$$

What does this twist look like? See the figure.

Exercise 1. show that T is not an isometry, but preserves area on the cylinder.

Now, let's consider a linear map on $\mathbb{R}^n$:

$$f : \mathbb{R}^n \to \mathbb{R}^n, \quad f(\vec{x}) = A\vec{x},$$

where A is a $n \times n$ matrix. Choose an orthonormal basis for $\mathbb{R}^n$. Then the standard cube C whose sides are the basis vectors will be mapped by f to a parallelepiped. What would be the volume of this image? Well, here

$$\mathrm{vol}\,(f(C)) = |\det A|.$$

What would be the conclusion one can draw from this? This is simply the infinitesimal version of any smooth map on $\mathbb{R}^n$. to see this, let's start with a better idea of what kind of sets have positive volume in $\mathbb{R}^n$. Recall in any metric space X, we can define a small open set via an inequality:

$$B_\epsilon(x) = \left\{ y \in X \,\middle|\, d(x, y) < \epsilon \right\}.$$

Definition 2. A subset $U \in X$ is called *open* if $\forall x \in U$, $\exists \epsilon > 0$ such that $B_\epsilon(x) \in U$. A subset is called closed if its complement is open.

Definition 3. A *domain* in X is either an open subset of X, or the closure of an open subset of X

This last definition ensures that a domain has non-zero volume, although the volume may be infinite. In $\mathbb{R}^n$ with the standard Euclidean metric, the ϵ-balls have volume $\frac{4}{3}\pi\epsilon^3 > 0$, when $\epsilon > 0$.

Proposition 4. *let $U \in \mathbb{R}^n$ be an open domain. A differentiable map $f : U \to \mathbb{R}^n$ preserves volume iff $|\det(Df_x)| = 1$, $\forall x \in U$.*

The Jacobian matrix of a function like f is the matrix of partial derivatives of f, and their values at a point $x \in U$ become the derivative matrix at that point Df_x. We sometimes refer to the determinant of this matrix the Jacobian of f, $Jac(f)$.

Definition 5. A map $f : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is a domain, preserves orientation if $\forall x \in U$, $Jac(f) > 0$.

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"Nice" ODEs (where solutions are uniquely defined, for example), are always orientation preserving. Recall the relationship between the time-1 map of a any linear ODE system on $\mathbb{R}^2$. It always had eigenvalues which were related to those of the original flow by the exponential map. Under the exponential map, the time-1 map will always have a positive Jacobian (why?).

More generally, let $\dot{\vec{x}} = \overrightarrow{f(\vec{x})}$ be an ODE on $\mathbb{R}^n$. Then the function f defines a vector field on $\mathbb{R}^n$ (to each point $\vec{x}$ we attach the vector $\overrightarrow{f(\vec{x})}$). Remember this fact about the divergence of the vector field from Calculus III?

Proposition 6. *If the divergence of the vector field f, $div(f) = 0$, then f preserves volume.*

Theorem 7. *let X be a finite volume domain in $\mathbb{R}^n$ or $\mathbb{T}^n$, and $f : X \to X$ be an invertible, volume preserving C^1-map. Then $\forall x \in X$ and $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that*

$$f^n\left(B_\epsilon(x)\right) \cap B_\epsilon(x) \neq \emptyset.$$

Proof. Really the cheap idea is this: Suppose $\exists x \in X$, and $\exists \epsilon > 0$ such that $\forall n \in \mathbb{N}$

$$f^n\left(B_\epsilon(x)\right) \cap B_\epsilon(x) = \emptyset.$$

Since f is volume preserving, we must have at the nth iterate:

$$\infty > \ \text{vol}(X) > \sum +i = 1^n \ \text{vol}\left(f^i\left(B_\epsilon(x)\right)\right) = n \cdot \ \text{vol}\left(B_\epsilon(x)\right).$$

But for all choice of $\epsilon > 0$,

$$\lim_{n\to\infty} n \cdot \ \text{vol}\left(B_\epsilon(x)\right) = \infty$$

since $\text{vol}\left(B_\epsilon(x)\right) > 0$. This contradiction establishes the proof. $\square$

This gives us an immediate consequence:

Corollary 8. *For $f : X \to X$ as above, $\forall x \in X$, there exists a sequence $\{y_k\} \longrightarrow x$ and a sequence $\{m_k\} \longrightarrow \infty$ where $\{f^{m_k}(y_k)\} \longrightarrow x$.*

See the figure or an idea of what is going on.

Exercise 2. Produce this sequence.

Definition 9. For $f : X \to X$ a continuous map of a metric space, a point $x \in X$ is called

- *positively recurrent* with respect to f if $\exists$ a sequence $\{n_k\} \longrightarrow \infty$ such that $\{f^{n_k}(x)\} \longrightarrow x$,
- if f is invertible, *negatively recurrent* if $\exists$ a sequence $\{n_k\} \longrightarrow -\infty$ such that $\{f^{n_k}(x)\} \longrightarrow x$,
- *recurrent* if it is both positively and negatively recurrent.

Definition 10. For $f : X \to X$ a continuous map of a metric space, the set

$$\omega(x) = \overline{\bigcap_{n\in\mathbb{N}} \left\{ f^i(x) \middle| i \geq n \right\}}$$

is the set of all accumulation points of the orbit of x. It is called the *ω-limit set* of $x \in X$ with respect to f. For f an invertible map on X, the set

$$\alpha(x) = \overline{\bigcap_{n\in-\mathbb{N}} \left\{ f^i(x) \middle| i \leq n \right\}}$$

is called the *α-limit set* of x with respect to f.

Note: $x \in X$ is positively recurrent if $x \in \omega(x)$ (if x is in its own ω-limit set).

Exercise 3. Show, by construction, that $\forall \alpha \in \mathbb{R}$, all points of S^1 are recurrent under the rotation map R_α.

Exercise 4. Show the same by construction for a translation on $\mathbb{T}^2$.

Theorem 11. *Let X be a closed finite-volume domain in $\mathbb{R}^n$ or $\mathbb{T}^n$ and $f : X \to X$ an invertible volume preserving map. Then the set of recurrent points for f is dense in X.*

Note; This does not mean that all points are recurrent, not that there may be tons of points whose ω-limit sets do not include the original point. It does mean that every point either is recurrent, or has a recurrent point arbitrarily close to it. We won't prove this here. The proof is in the book on page 160. Instead, let's skip ahead to Section 6.2.

2. Newtonian Systems of Classical Mechanics

Your previous work in ODEs suggested a general premise about systems of differential equations. If they are defined "nicely", then the present state of a mechanical system determines its future evolution through other states uniquely. One can place this in the language of dynamical systems to say that if a mathematical construction accurately models a mechanical system, than the construction determines a dynamical system on the space of all possible states of the system. The trick in many cases is to well understand what constitutes a state of a mechanical system. To start, given a mechanical system, the *configuration space* of the system is the set of all possible positions (value combinations of all of its variables) of the system. The *state space*, rather, is the set of all possible states the system can be in. This is usually much broader a description.

For example, consider the pendulum, a mass is attached to the free end of a massless rigid rod, while the other end of the rod is fixed. The set of all possible configurations of the pendulum is simply a copy of S^1. However, for each configuration, the pendulum is in a different state depending on what the mass' velocity is when it resides in a configuration. One can think of all possible states as the space $S^1 \times \mathbb{R}$. This reflects the data necessary to completely determine the future evolution of the pendulum by a knowledge of its position and velocity at a single moment, and the evolution equation which is a second-order non-linear ODE in the general form

$$\ddot{x} = f(t, x, \dot{x}),$$

where in this case (and many others), time is not explicit on the right hand side. Under the standard practice of converting this ODE into a system of two first order ODEs, we can interpret the evolution as giving a vector field on the state space $S^1 \times \mathbb{R}$, with coordinates x and $\dot{x}$. This vector field determines a flow, which solves the ODE and determines the future evolution of the system based on knowledge of a particular moment's data.

Many systems behave in a way that their future states are completely determined by their present position and velocity, along with a notion of how they are changing. In classical (Newtonian) mechanics, Newton's Second Law of motion states roughly that the force acting on an object is proportional to how the velocity of the object is changing. The is the famous equation $f = ma$, where f is the total force acting on the object and a is its acceleration. As the velocity depends on the current position of an object, a good notion of how an object moves through a space under the influence of a force is completely determined bo how its position and velocity are changing, at least when the force is static:

$$f(x) = ma = m\ddot{x} = m\frac{d^2x}{dt^2}.$$

This is a special case of the general second order ODE mentioned above.

Next class, we will continue with a few examples of such systems before we analyze the structure these systems have.