MATH 421 DYNAMICS

Week 7 Lecture 2 Notes

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1. Invertible S^1-maps (cont'd.)

Last class, we established some of the properties of the lifts of circle maps. We continue this now. Again, let $f : S^1 \to S^1$ be a circle map, and $F : \mathbb{R} \to \mathbb{R}$ a lift to $\mathbb{R}$, defined as a map which satisfies the criterion that $f \circ \pi(x) = \pi \circ F(x)\ \forall x \in \mathbb{R}$, where $\pi : \mathbb{R} \to S^1$ is the standard projection given by the exponential map, $\pi(x) = e^{2\pi i x}$, also denoted $\pi(x) = [x]$.

- *if f is a homeomorphism, then $|\deg(f)| = 1$.*

  Proof. Suppose that $|\deg(f)| > 1$. Then $|F(x+1) - F(x)| > 1$. And since $F(x+1) - F(x)$ is continuous, by the Intermediate Value Theorem, $\exists y \in (x, x+1)$ where $|F(y) - F(x)| = 1$. But then $f([y]) = f([x])$ for some $y \neq x$. Thus f cannot be one-to-one and hence cannot be a homeomorphism.

  Now suppose that $|\deg(f)| = 0$. Then $F(x+1) = F(x)$, $\forall x$, and hence F is not one-to-one on the interval $(x, x+1)$. But then neither is f, and again, f cannot be a homeomorphism. □

- *$F(x) - x\deg(f)$ is periodic.*

  Proof. It is certainly continuous (why?) To see that it is periodic (of period-1), simply evaluate this function at $x+1$:
  $$F(x+1) - (x+1)\deg(f) = (F(x) + \deg(f)) - (x+1)\deg(f) = F(x) - x\deg(f).$$
  □

Example 1. Let $f(x) = x$. This is the "identity" map on S^1, since all points are taken to themselves. A suitable lift for f is the map $F(x) = x$ on $\mathbb{R}$. to see this, make sure the definition works. Question: Are there any other lifts for f? What about the map $\overline{F}(x) = x + a$ for a a constant? Are there any restrictions on the constant a? The answer is yes. For a to be an acceptable constant, we would need the definition of a lift of be satisfied. Thus
$$\left[\overline{F}(x)\right] = [x + a] = f([x]) = [x].$$
So the condition that a must satisfy is $[x+a] = [x]$ on S^1. Hence, $a \in \mathbb{Z}$. A new question: For a real number $a \notin \mathbb{Z}$, can $\overline{F}(x) = x + a$ serve as a lift of a circle map? What sort of circle map?

Example 2. Let $f(x) = x^n$. Thinking of x as the complex number $x = e^{2\pi i\theta}$, for $\theta \in \mathbb{R}$, then
$$f(x) = f\left(e^{2\pi i\theta}\right) = \left(e^{2\pi i\theta}\right)^n = e^{2\pi i(n\theta)}.$$
Hence a suitable lift is obviously $F(x) = nx$ (I say obviously, since the variable in the exponent <u>is</u> the lifted variable!) Question: This is a degree n map. For which values of n does the map f have an inverse" And what does the map f actually do for different values of n?

Example 3. Let f be a general degree-r map. Then $F(1) - F(0) = r = deg(f)$. Suppose that $F(0) = 0$. Then $F(1) = r$ and if, for example, $r > 1$, it is now easy to see that there will certainly be a $y \in (0, 1)$, where $F(y) = 1$. This was a fact that we used in the proof above to show that f cannot be a homeomorphism. In this case, where $r > 1$, at every point in $y \in (0, 1)$ where $F(y) \in \mathbb{Z}$, we will have $\pi \circ F(y) = [F(y)] = 0$ on S^1. This won't happen when $r = 1$. When $r = 0$, the map F will be periodic, which is definitely not one-to-one. Question: What happens when $r < 0$? Draw some representative examples to see.

Definition 4. Suppose that $f : S^1 \to S^1$ is invertible. Then

(1) if $\deg(f) = 1$, f is orientation preserving.
(2) if $\deg(f) = -1$, f is orientation reversing.

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Recall from Calculus III that orientation is a choice of direction in the parameterization of a space (really, it exists outside of any choice of coordinates on a space, but once you put coordinates on a space, you have essentially chosen an orientation for that space. This is true at least for those spaces that actually are orientable, that is (Mo bius Band?) On R, it is the choice of direction for the symbol "¿". On a surface, it is a choice of side. In $\mathbb{R}^3$, one can use the Right Hand Rule. Etc. On S^1, orientation preserving really means that after one applies the map, points to the right of a designated point remain on that side. Orientation reversing will flip a very small neighborhood of a point.

Circle maps may or may not have periodic points. And given an arbitrary homeomorphism, without regard to any other specific properties of the map, one would expect that we can construct maps with lots of periodic points of any period. However, it turns out that circle homeomorphisms are quite restricted. because they must be one-to-one and onto, only certain things can happen. To explain, we will need another property of circle homeomorphisms to help us.

Proposition 5. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism, with $F : \mathbb{R} \to \mathbb{R}$ a lift. Then the quantity*

$$\rho(F) := \lim_{|n|\to\infty} \frac{F^n(x) - x}{n}$$

(1) exists $\forall x \in \mathbb{R}$,
(2) is independent of the choice of x and is defined up to an additive integer, and
(3) is rational iff f has a periodic point.

Given these qualities, the additional quantity $\rho(f) = [\rho(F)]$ is well-defined and is called the *rotation number* of f.

Some notes:

- $\rho(R_\alpha) = \alpha$. (You should be able to actually calculate this using the definition. Do it.)
- ρ represents in a way the average rotation of points in a circle homeomorphism.
- **Proposition 6.** *If $\rho(f) = 0$, then f has a fixed point.*
  Another way of saying that if there is no average rotation of the circle map, then somewhere a point doesn't move under f. This is like the Intermediate value Theorem on a closed, bounded interval of $\mathbb{R}$ where a map is positive at one end point and negative at the other.
- If f has a q-periodic point, then for a lift F, we have $F^q(x) = x + p$ for some $p \in \mathbb{Z}$. For example, let $f = R_{\frac{6}{7}}$. Then a suitable lift for f would necessarily satisfy $F^7(x) = x + 6$, ∀x∈ $\mathbb{R}$. Question: Write down two such lifts for this choice of map f.
- **Proposition 7.** *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism. Then all periodic points must have the same period.*

This last point is quite restrictive. Essentially, if an orientation preserving homeomorphism has a fixed point, it cannot have periodic points of any other period, say. Note that this is not true of a orientation reversing map. For example, the map which flips the unit circle in $\mathbb{R}^2$ across the y-axis, will fix the two points $(0, 1)$ and $(0, -1)$, while every other point is of order two.

This is enough for circle homeomorphisms for now. And ends our work in Chapter 4. There is a great section on frequency locking on page 141. Look it over at your leisure. We won't work through it in the course, but it is very interesting. Dynamically, it represents a situation where a linear flow on the torus (with its uncoupled ODEs) becomes the limiting system to a system of coupled ODEs, representing a nonlinear flow. Question: For this to be the case, must the resulting linear flow on the torus be a rational flow?

2. Chapter 5

really, in this short chapter, the onely thing I want to discuss is a way to understand toral flows in higher dimensions. For this, let's describe the space. By definition, the n-dimensional torus, or the n=torus, denoted $\mathbb{T}^n$ is simply the n-fold product of n circles

$$\mathbb{T}^n = \overbrace{S^1 \times \cdots \times S^1}^{n \text{ times}}.$$

Think of a system of equations where the n variables are all angular coordinates. Then

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}.$$

Recall the Kepler Problem. With n point masses, the resulting flow may be seen as linear motion on $\mathbb{T}^n$.

Another way to view the n-torus is via an identification within $\mathbb{R}^n$. Remember the unit square with it opposite sides identified plays a good model for the 2-torus, $\mathbb{T} = \mathbb{T}^2$. The generalization works well here for all the natural numbers. Take the unit cube in $\mathbb{R}^3$. Identify each of the opposite pairs (think of a die, and identify two sides if their numbers add up to 7). The resulting model is precisely the $\mathbb{T}^3$. This works well if one wants to watch a flow on $\mathbb{T}^3$. Simply allow the flow to progress in the unit cube, and whenever one hits a wall, simply vanish and reappear on the opposite wall, entering back into the cube.

Note this also works well for $n = 1$: Take the unit interval and identify its two sides (the numbers $x = 0$ and $x = 1$). This is what I mean by the phrase $0 = 1$ on S^1, where the circle is the 1-torus.

Now, the vector exponential map

$$(\theta_1, \ldots, \theta_n) \overset{\exp}{\longmapsto} \left(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}\right)$$

maps $\mathbb{R}^n$ onto $\mathbb{T}^n$. We can define a (vector) rotation on $\mathbb{T}^n$ by the vector $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where

$$R_{\vec{\alpha}}\left(\vec{x}\right) - (x_1 + \alpha_1, \ldots, x_n + \alpha_1) = \vec{x} + \vec{\alpha}.$$

Note that it should be obvious that if all of the α_i's are rational, then the resulting flow on $\mathbb{T}^n$ whose coordinate flows are $x_i(t) = x_i + \alpha_i t$ will have closed orbits. The question is, are theses the only periodic linear flows? We saw how it was the ratio of the two flow rates that determined whether the flow had closed orbits on $\mathbb{T}^2$. But how do we define ratios in higher dimensions? By a notion of rational independence:

Definition 8. A set of n real numbers $\{\alpha_i\}_{i=1}^n$ is said to be *rationally independent* if there are no non-zero integer solutions $\vec{k} = (k_0, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1}$ to

$$k_0 + k_1\alpha_1 + \ldots + k_n\alpha_n = 0.$$

Another way to say this is the following: There does not exist an integer vector $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ where

$$\sum_{n=1}^{n} k_i \alpha_i = \vec{k} \cdot \vec{\alpha} \in \mathbb{Z},$$

unless ALL of the k_i's are simultaneously 0. We have the following:

Proposition 9. *The linear flow on $\mathbb{T}^n$ whose time-1 map is the rotation $R_{\vec{\alpha}}$ is minimal iff the numbers $\alpha_1, \dots, alpha_n, 1$ are rationally independent.*

Example 10. On the two torus, let $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \pi$. The flow will be minimal here since the numbers 1, $\frac{1}{2}$ and $[\pi]$ are rationally independent:

$$k_0 + \frac{k_1}{2} + k_2\pi = 0 \iff \pi = \frac{2k_0 + k_1}{2k_2} \in \mathbb{Q},$$

at least for $k_0, k_1, k_2 \in \mathbb{Z}$. On the other hand, let $\alpha_1 = 2\pi$ and $\alpha_2 = \frac{pi}{6}$. Since the ratio of the α's is rational, the flow is not minimal (all orbits are closed here. Given this definition, we would get rational dependence:

$$k_0 + 2\pi k_1 + \frac{k_2\pi}{6} = 0 \iff k_0 = -\frac{12k_1 + k_2}{6}\pi.$$

While this may look impossible to solve for three integers k_0, k_1, and k_2, the only stipulation is that not all of the integers can be 0. But some of them can! Simply allow $k_0 = 0$. Then choose any non-zero integer as your k_1, and let $k_2 = -12k_1$.

3. Chapter 6: Conservative Systems

In Chapter 4, we first looked at what was considered "recurrent" behavior, which roughly means that the orbit of a point, passes arbitrarily close to the point. This worked well in the classification of circle rotations, since either the orbit of a point was closed (the orbit was periodic; the rotation was rational) or the orbit was dense (for an irrational flow). In either case, and also for toral flows, every point was recurrent.

Contrast this with the dynamical systems that we studied in Chapters 2 and 3. Here, with examples like contracting maps and sinks and sources, the only recurrent points were the fixed and periodic points, and there were very few of those in each system. More generally, maps can exhibit much more complicated behavior. To understand this behavior, we will have to broaden our idea of how to study such systems. This chapter begins this study.

To start, let's change our perspective. Given a dynamical system, let's not worry about how an individual orbit behaves so much as how a whole families of nearby orbits evolve. This would be more like following all of the orbits that start in a small neighborhood over the evolution of the map. For a contraction, this would be easy and not very insightful. (why?) But for a general map, this idea can be quite interesting.

4. Incompressibility

Also called phase volume preservation, Suppose as one evolves via a flow, or iterates via a map, that the volume of a small domain does not change. Then the volume is said to be preserved by the flow (map), or the volume is invariant under the flow (map). Obvious examples include linear flows in $\mathbb{R}^n$, rotation maps on S^1 (remember that volume in a space like $\mathbb{R}$ or S^1 is just length, and in dimension 2 is just area), and linear toral flows. Examples which do not preserve volume include contraction maps, and flows (defined by ODEs) that include sinks and sources (saddles, maybe, though).

In fact, if the map is by isometries (or the flow has all of its time-t map given by isometries), then the volume will be preserved. this should be obvious, as if all of the distances between the points of a small domain are preserved, the volume cannot change. The converse is not true however. Lots of maps and flows preserve volume but are not isometries. This we will see to start the next class.