MATH 421 DYNAMICS

Week 6 Lecture 2 Notes

Linear flows on the 2-torus (cont'd.)

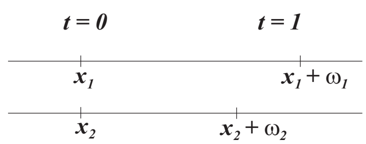
Given $S^1 = \left\{e^{2\pi i x} \in \mathbb{C}\right\}$, and $\frac{dx}{dt} = \alpha$, $x(0) = x_0$ an IVP defined on S^1, we saw last time that even though the flow $\varphi^t_\alpha(x) = \alpha t + x$ had as its time-1 map

$$\varphi^1_\alpha(x) = \alpha + x = R_\alpha(x), \quad x \in S^1$$

a rotation of the circle, the continuous flow was not interesting dynamically. However, we can generalize this flow to a situation which does produce somewhat interesting dynamics.

Consider now a flow given by the pair of uncoupled circle ODEs:

$$\frac{dx_1}{dt} = \omega_1, \quad \frac{dx_2}{dt} = \omega_2.$$



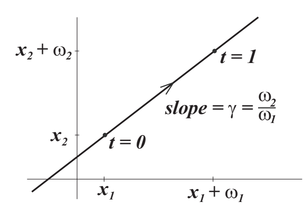
This system, which can be written as the uncoupled vector ODE $\dot{\overrightarrow{x}} = \overrightarrow{\omega}$, or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$, can be viewed as defining a flow on the two-torus $\mathbb{T} = S^1 \times S^1$, and has the solution

$$\vec{x} = \begin{bmatrix} x_1 + \omega_1 t \\ x_2 + \omega_2 t \end{bmatrix}.$$

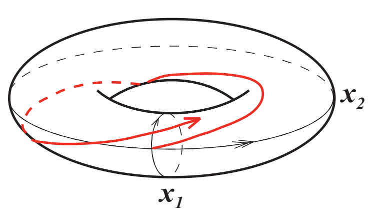
In flow notation, we can write either

$$T^t_{\overrightarrow{\omega}}(x_1, x_2) = (x_1 + \omega_1 t, x_2 + \omega_2 t)\,, \text{ or } \varphi^t_{\overrightarrow{\omega}}(\overrightarrow{x}) = \overrightarrow{x} + \overrightarrow{\omega} t.$$

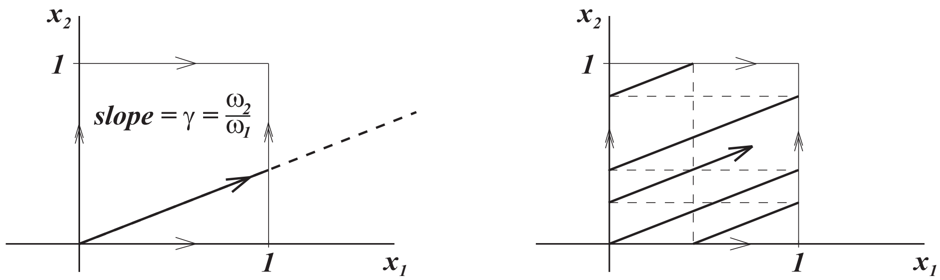
Graphically, solutions are simply translations along $\mathbb{R}$ or as straight line motion in $\mathbb{R}^2$. Note that in this last interpretation, the slope of the solution line is $\gamma = \frac{\omega_2}{\omega_1}$.



However, each of these uncoupled ODEs also can be considered as a flow on S^1, and hence the system can be considered a flow on $S^1 \times S^1 = \mathbb{T}$. Suppose, for example, that $1 < \omega_1 < 2$, while $0 < \omega_2 < 1$. The flow from time $t = 0$ to time $t = 1$ would take the origin on one circle to the point $1 - \omega_1$, and the flow line would start at $x_1 = 0$ and travel once around the circle before stopping to ω_1. The flow on the other circle would take $x_2 = 0$ partway around the circle to ω_2. Viewed via the two periodic coordinates of $\mathbb{T}$, we have the flow line in the picture:



Another way to see this is to go back to the plane and consider the equivalence relation given by the exponential map on each coordinate. The set of equivalence classes are given by the unit square in the plane, under the idea that the left side of the square (the side on the $x_1 = 0$-line) and the right side (the $x_1 = 1$-side) are considered the same points (this is the $0 = 1$ idea of the circle identification). Similarly, the top and bottom of the square are to be identified. Then the flow line at the origin under the ODE system is a straight line of slope γ emanating from the origin and meeting the right edge of the unit square at the point $(1, \gamma)$. But by the identification, we can restart the graph of the line at the same height on the left side of the square (at the point $(0, \gamma)$. Continuing to do this, we will eventually reach the top of the square. But by the identification again, we will drop to the bottom point and continue the line as before. In essence, we are graphing the flow



line as it would appear on the unit square. When we pull this square out of the plane and bend it to create our torus $\mathbb{T}$, the flow line will come with it. Suppose $\gamma \notin \mathbb{Q}$. What can we say about the positive flow line?

Proposition 1. *if $\gamma = \frac{\omega_2}{\omega_1}$ is irrational, then the flow is minimal. If $\gamma \in \mathbb{Q}$, then every orbit is closed.*

Proof idea. Choose any circle (an easy-to-see choice would be the $x_1 = 0$ circle, which is represented in the plane by the left edge of the unit square). We could call this the waist circle. Then for any point $(0, y)$ on the waist circle, the first-return map of $\mathcal{O}^+_{(0,y)}$ is exactly the rotation map of this circle given by R_γ. As the entire waist circle flows around $\mathbb{T}$, and with the irrational rotation, the orbit $\mathcal{O}^+_{(0,y)}$ will intersect the waist circle densely, the orbit of $(0, y)$ (and hence every point) will be dense in all of $\mathbb{T}$.

For the other statement, simply show that the orbit of every point will eventually return to its starting point, and since the flow is always along straight lines, this is enough to show the periodicity of any and hence all points. ∎

Example 2. We can look at this another way: Think of $S^1 \in \mathbb{R}^2$ as a circle of radius r centered at the origin. Then we can represent $\mathbb{T}$ as the set

$$\mathbb{T} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,\middle|\, x_1^2 + x_2^2 = r_1^2, \;\; x_3^2 + x_4^2 = r_2^2 \right\}.$$

Now recall a continuous rotation in $\mathbb{R}^2$ is given by the linear ODE system $\dot{\vec{x}} = B_\alpha \vec{x}$, where B is the matrix $\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$ whose eigenvalues $\pm\alpha i$ are purely imaginary. Do this for each pair of coordinates (each of two copies of $\mathbb{R}^2$) to get the partially uncoupled system of ODEs on $\mathbb{R}^4$,

$$\dot{\vec{x}} = A\vec{x}, \qquad \begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 \\ -\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & -\alpha_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

We will eventually see that this is the model for the spherical pendulum.

Some notes:

- The two circles $x_1^2 + x_2^2 = r_1^2$ and $x_3^2 + x_4^2 = r_2^2$ are invariant under this flow.
- We can define angular coordinates on $\mathbb{T}$ via the equations

$$x_1 = r_1 \cos 2\pi\varphi_1 \quad x_2 = r_1 \sin 2\pi\varphi_1$$
$$x_3 = r_2 \cos 2\pi\varphi_2 \quad x_4 = r_2 \sin 2\pi\varphi_2.$$

Date: March 15, 2010.

Then, restricted to these angular coordinates and with $\omega_i = \frac{\alpha_i}{2\pi}$, $i = 1, 2$, we recover

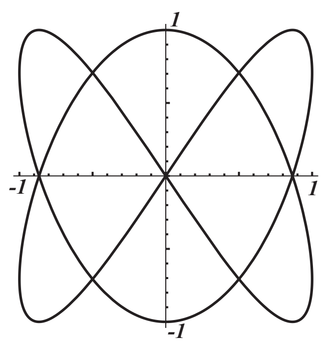
$$\dot{\varphi}_1 = -\omega_1, \quad \dot{\varphi}_2 = -\omega_2.$$

Motion is independent along each circle, and the solutions are $\varphi_i(t) = \omega_i(t - t_0)$.

- If $\frac{\alpha_2}{\alpha_1} = \frac{\omega_2}{\omega_1} \notin \mathbb{Q}$, then the flow is minimal.

Exercise 1. Do the change of coordinates explicitly to show that these two interpretations of linear toral flows are the same.

Now, for a choice of $\overrightarrow{\omega}$ and $r_1 = r_2 = 1$, project a solution onto either the (x_1, x_3) or the (x_2, x_4)-planes. The resulting figure is a plot of a parameterized curve whose two coordinate functions are cosine (resp. sine) functions of periods which are rationally dependent iff $\overrightarrow{\omega}$ is rational. In this case, the figure is closed, and is called a Lissajous figure. See the figure below for the case of two sine functions (projection onto the (x_2, x_4)-plane, in this case), where $\omega_1 = 2$ and $\omega_2 = 3$.



$\omega_1 = 2, \quad \omega_2 = 3$

Q. What would the figure look like if ω_1 and ω_2 were not rational multiples of each other?

A nice physical interpretation of this curve is as the trajectory of a pair of uncoupled harmonic oscillators, given by

$$\begin{aligned} \ddot{x_1} &= -\omega_1 x_1 \\ \ddot{x_2} &= -\omega_2 x_2. \end{aligned}$$

Next class, we will begin a study of a related type of dynamical system called a billiard. In one of its most elementary forms, the model of straight-line motion in the plane and its corresponding linear flow on a 2-torus again appear.