

MATH 421 DYNAMICS

Week 6 Lecture 1 Notes

ROTATIONS OF THE CIRCLE (CONT'D.)

We ended last time with a discussion of what density means for the orbit of a point in S^1 under the irrational rotation R_α . We were working on the example:

Example 1. So let R_α be a rotation of S^1 for $\alpha = \frac{1}{3 + \frac{1}{5 + \frac{1}{c}}}$, where $c > 1$, and $c \notin \mathbb{Q}$. Then it turns out that $\alpha \notin \mathbb{Q}$. Let's start the construction which would establish the middle bullet point in the above proof idea. To start, it should be obvious that $\frac{1}{4} < \alpha < \frac{1}{3}$ (why?). Denote, as in the figure, $a = R_\alpha(0)$, $b = R_\alpha^2(0)$, $c = R_\alpha^3(0)$, and $d = R_\alpha^4(0)$. One of c and d winds up being the early, closest approach to 0 of the orbit $\mathcal{O}_0^+$. But which is smaller, $\delta = 1 - 3\alpha$, or $\beta = 4\alpha - 1$? Well, the first calculation can be performed via the partial CFR given here:

$$\begin{aligned} \delta = 1 - 3\alpha &= 1 - \frac{3}{3 + \frac{1}{5 + \frac{1}{c}}} = \frac{3 + \frac{1}{5 + \frac{1}{c}} - 3}{3 + \frac{1}{5 + \frac{1}{c}}} \\ &= \frac{\frac{1}{5 + \frac{1}{c}}}{3 + \frac{1}{5 + \frac{1}{c}}} = \frac{1}{16 + \frac{3}{c}}. \end{aligned}$$

I ask you to calculate β in the exercises, and show that it is greater than δ . Hence, the third iterate is the first closest return of $\mathcal{O}_0^+$ to 0).

- Q.** Will the orbit ever get closer to 0?
- Q.** If it will, then which iterate?

These questions will be necessary to prove that the orbit will eventually get arbitrarily close to 0.

We could simply hunt for the next return. Or we can be clever and calculate it. Here is the idea: it took three steps to get within δ of the initial point 0. (We could say it took three steps to get δ -close to 0). If we now create an open δ -neighborhood of 0, $N_\delta(0)$, when will the first iterate occur when we will enter this neighborhood and thus get closer than δ to 0?

One way to ensure this is to look at the first step after our previous close approach. This is the fourth element of $\mathcal{O}_0$ and is $R_\alpha^4(0) = 4\alpha$. Here $4\alpha = \alpha + 3\alpha = \alpha + (1 - \delta)$, so that $4\alpha - 1 = \beta = \alpha - \delta$. One conclusion to draw from this is that $R_{3\alpha}$ takes α to 4α which is $\alpha - \delta$ (see figure). So $R_{3\alpha}(\alpha) = \alpha - \delta$, $R_{3\alpha}^2(\alpha) = \alpha - 2\delta$, and $R_{3\alpha}^n(\alpha) = \alpha - n\delta$. So for which n would we satisfy

$$0 < \alpha - n\delta < \delta?$$

Well, this simplifies to $n\delta < \alpha < (n+1)\delta$, which is solved by simply taking the integer part of the fraction $\frac{\alpha}{\delta}$. Denote the greatest integer function by $\lfloor \cdot \rfloor$, so that, for example, $\lfloor \pi \rfloor = 3$. Then, the iterate n we are looking for is

$$\lfloor n \rfloor = \left\lfloor \frac{\alpha}{\delta} \right\rfloor = \left\lfloor \frac{\frac{1}{3 + \frac{1}{5 + \frac{1}{c}}}}{\frac{1}{16 + \frac{3}{c}}} \right\rfloor = \left\lfloor 5 + \frac{1}{c} \right\rfloor = 5.$$

Hence we can say that $R_{3\alpha}^5(\alpha) = R_\alpha^{15}(\alpha) = R_\alpha^{16}(0)$ is within δ of 0. We could then use the actual distance between 0 and $R_\alpha^{16}(0)$ as our new δ , and look for iterates of R_α^{16} to find our next closest approach. Continuing this way, we create a subsequence of $\mathcal{O}_0$ which consists of exponentially increasing powers of the original R_α

and this subsequence converges to 0. This is the basic approach to proving the second bullet point in the above proof idea.

On the real line, we see that our rotations by α is simply a translation by α . Approaching and getting closer to 0, means that our orbit will at some point come close to an integer value (ANY integer will do, as they all represent 0 in the circle!). See the figure.

There is really a better way to understand this notion of visiting neighborhoods of points in S^1 under irrational rotations. This other way is by understanding the frequency with which an orbit visits a small open set under a rotation. This is called the *dynamics frequency*, and is a measure of how often an orbit visits a small open interval in S^1 relative to how much time it is outside of the interval.

Fix $\Delta \subset S^1$ an arc. Then for $x \in S^1$ and $n \in \mathbb{N}$, define

$$F_\Delta(x, n) = \# \{ k \in \mathbb{Z} \mid 0 \leq k < n, R_\alpha^k(x) \in \Delta \}.$$

Here, the number sign $\#$ denotes the cardinality of the set.

Note that for Δ small, then for any $x \in S^1$, F_Δ will be small. And for Δ large, F_Δ will be bigger, but always less than n . So we can say that $0 \leq F_\Delta(x, n) \leq n$, for every x and Δ . And for any choice of x and Δ , as n grows, F_Δ is monotonically increasing.

However, it is also true that for $\alpha \notin \mathbb{Q}$,

$$\lim_{n \rightarrow \infty} F_\Delta(x, n) = \infty.$$

Hence instead of studying the frequency with which the orbit of a point visits an arc, we study the *relative frequency* of visits as n gets large, or the quantity

$$\frac{F_\Delta(x, n)}{n}.$$

Suppose on the orbit segment of a point x under the irrational rotation by α given by $\{x = R_\alpha^0(x), R_\alpha^1(x), \dots, R_\alpha^m(x)\}$, we found that given the arc Δ , that $R_\alpha^{k_1}(x), R_\alpha^{k_2}(x), R_\alpha^{k_3}(x) \subset \Delta$ and these were the only three. Then we know that the frequency $F_\Delta(x, m) = 3$, and the relative frequency $\frac{F_\Delta(x, m)}{m} = \frac{3}{m}$ (see the figure). The goal is to study the relative frequency of a rotation on any arc of any length and be able to say something meaningful about how often it visits the arc.

Some notes:

- Define $\ell(\Delta) = \text{length of } \Delta$ (under some metric).
- The relative frequency really does not depend on whether Δ is open, closed or neither (why not?).
- The convention is to take representatives for arcs to be of the “lqs half-closed” form $[\cdot, \cdot)$. Then it is easy to see whether unions of arcs are connected or not.
- here we will actually study the overall relative frequency of entire orbits: This translates to a study of

$$\lim_{n \rightarrow \infty} \frac{F_\Delta(x, n)}{n}.$$

However, It is yet not entirely clear that this quantity actually exists. We first have to endure this:

This last point needs some elaboration. Recall the function $f : \mathbb{N} \rightarrow \mathbb{R}$ defined by $f(n) = \sin n$. Here, we know that $\lim_{n \rightarrow \infty} f(n)$ does not exist (recall the continuous version of this function in calculus. There is no horizontal asymptote for f). So we first define the limit inferior (respectively superior) for f . This type of limit either always exists or is $-\infty$ (resp. ∞). It is the largest (resp. smallest) number where no more than

a finite number of terms in the sequence are smaller (resp. larger) than it on the entire sequence. Think of the envelope of a sequence being defined to allow some terms to be outside the envelope, but only a finite number of them. In the case of $f(n) = \sin n$, the $\liminf_{n \rightarrow \infty} f(n) = -1$. This makes sense, since if we try to “cut” the function at anything above -1 , that small interval of values (think $[-1, -1 + \epsilon)$) will be visited an infinite number of times eventually by f . Also, $\limsup_{n \rightarrow \infty} f(n) = 1$. It should be obvious that while these quantities may not be easy to calculate, not only should they exist (for the minute, think of an infinite limit as existing in the sense that the sequence is going somewhere), but it must be the case that for any sequence $\{x_n\}_{n \in \mathbb{N}}$,

$$\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n.$$

And should they be equal, then $\lim_{n \rightarrow \infty} x_n$ in fact exists and is equal to the two limit bounds.

In our case, let A be a disjoint union of arcs. Then define

$$\bar{f}_x(A) = \limsup_{n \rightarrow \infty} \frac{F_A(x, n)}{n}, \quad \underline{f}_x(A) = \liminf_{n \rightarrow \infty} \frac{F_A(x, n)}{n}.$$

It turns out that these two quantities not only exist. They also are equal:

Proposition 2. *For any arc $\Delta \subset S^1$, and every $x \in S^1$, and any irrational rotation R_α , $\alpha \notin \mathbb{Q}$ on S^1 , we have*

$$f(\delta) := \lim_{n \rightarrow \infty} \frac{F_\Delta(x, n)}{n} = \ell(\Delta).$$

idea. The proof relies on finding bounds for the quantities $\bar{f}_x(\Delta)$ and $\underline{f}_x(\Delta)$, and showing that it is always the case that $\bar{f}_x(\Delta) \leq \ell(\Delta)$ and $\underline{f}_x(\Delta) \geq \ell(\Delta)$. This can only be the case if the limits superior and inferior are in fact equal, and equal to $\ell(\Delta)$. $\square$

Notes: Let $R_\alpha : S^1 \rightarrow S^1$ be an irrational rotation. Then for $x \in S^1$,

- the property of the sequence $\{a_n\}_{n \in \mathbb{N}}$, where $a_n = R_\alpha^n x$, is called a *uniform distribution* or an *equidistribution* on S^1 .
- the orbit $\mathcal{O}_x$ in a sense “fills” every arc in S^1 .

Hence, we say that any orbit of an irrational rotation of S^1 is uniformly distributed on S^1 .

We will need a bit more nomenclature to accurately describe this new kind of dynamical behavior:

Definition 3. A set $Y \subset X$ is invariant under a map $f : X \rightarrow X$, if

$$f \Big|_Y : Y \rightarrow Y.$$

Definition 4. A homeomorphism $f : X \rightarrow X$ is called *topologically transitive* if $\exists x \in X$ such that $\mathcal{O}_x$ is dense in X . An non-invertible map is called topologically transitive if $\exists x \in X$ such that $\mathcal{O}_x^+$ is dense in X .

Definition 5. A homeomorphism $f : X \rightarrow X$ is *minimal* if $\forall x \in X$ $\mathcal{O}_x$ is dense in X (the forward orbit is dense for a noninvertible map).

Definition 6. A closed, invariant set is *minimal* if there does not exist a proper, closed invariant subset.

More notes:

- Like in the case of open and closed domains in vector calculus, a set is closed if it contains all of its limit points. And for any set X , the closure of X , denoted $\overline{X}$ is defined to be the closed set obtained by adding to X all of its limit points (think of adding the sphere which is the boundary of an open ball in $\mathbb{R}^3$). In the case of a minimal map $f : X \rightarrow X$, for any $x \in X$, we have $\overline{\mathcal{O}_x} = X$.
- Same is true for a topologically transitive map f , if one takes any point on the dense orbit.
- Irrational rotations of the circle are minimal!

Here is a nice application:

Consider the two functions in the picture.

- Each is periodic.
- Each can be viewed as a real-valued smooth function on S^1 . And each takes values in the interval $I = [-1, 1]$.
- Question: Are the values of these two functions equally distributed equally (or even evenly) on I ?

Dynamics attempts to answer this question. Let $\{x_n\}$ be a sequence (think of this sequence as a sampling of the function), and $a < b$ two real numbers. Define

$$F_{a,b}(n) = \# \left\{ k \in \mathbb{Z} \mid i \leq k \leq n, a < x_k < b \right\}.$$

Really, this is the same definition of F as before on the arc Δ . This only change is that we are defining F in this context as an interval in $\mathbb{R}$. Here, $\{x_n\}$ has an *asymptotic distribution* if $\forall a, b$, where $-\infty \leq a < b \leq \infty$, the quantity

$$\lim_{n \rightarrow \infty} \frac{F_{a,b}(n)}{n}$$

exists.

In the case where the sequence has an asymptotic distribution, the function

$$\Phi_{\{x_n\}}(t) = \lim_{n \rightarrow \infty} \frac{F_{-\infty,t}(n)}{n}$$

is called the distribution function of the sequence $\{x_n\}$. Here Φ is monotonic, and measures how often the values of a sequence visit regions of the real line.

Definition 7. A real-valued function φ on an interval is called *piecewise monotone* if the domain can be partitioned into finite many intervals on which φ is monotonic.

Remark 8. Really, this means that there are no flat (purely horizontal on an open interval) regions of the graph of φ . Think of functions like $f(x) = \sin x$, and polynomials of degree larger than 1, which are piecewise monotonic, and functions like

$$g(x) = \begin{cases} -(x+2)^2 & -4 \leq x < -2 \\ 0 & -2 \leq x \leq 0 \\ x^2 & 0 < x \leq 2 \end{cases},$$

which is not piecewise monotonic (graph this to see it).

When φ is piecewise monotonic, the pre-image of any interval I is a finite union of intervals in the domain (see the figure).

Definition 9. The φ -length of an interval I is

$$\ell_\varphi(I) := \ell(\varphi^{-1}(I)).$$

- This is the total length of all pieces of the domain that map onto I . In the figure, $\ell_\varphi(I) = \ell(A) + \ell(B)$.
- For piecewise monotonic functions φ , the φ -length is a continuous function of the end points of I (vary one end point of I continuously, and the φ -length of I also varies continuously. This doesn't work with flat regions since the φ -length ℓ_φ would then jump as one hits the value of the flat region.

Example 10. Indeed, let's look at the $g(x)$ in the figure more closely. Here, one can calculate the φ -length. Indeed, choose the interval $I = [-4, t]$. Here, t is the function value, and there is only a single interval mapped onto i for any value of t .

For $t < 0$, this interval is given in the figure as the interval of the domain $g^{-1}(I) = [-4, r]$, where $g(r) = t$. Solving the equation $g(r) = t$ for r yields

$$-(r+2)^2 = t \iff r+2 = -\sqrt{-t} \iff r = -\sqrt{-t} - 2$$

where we chose the negative branch of the square root function in the middle step to account for the domain restrictions. Here, the g -length of I ,

$$\begin{aligned} \ell_g(I) &= \ell(g^{-1}([-4, t])) \\ &= -2 - \sqrt{-t} - (-4) = 2 - \sqrt{-t}. \end{aligned}$$

Now for $t > 0$, the same calculation yields $\ell_g(I) = 4 + \sqrt{t}$ for $I = [-4, t]$. Putting these two pieces of the g -length function together yields the graph of

$$\ell_g(I) = \begin{cases} 2 - \sqrt{-t} & -4 \leq t < 0 \\ 4 + \sqrt{t} & 0 < t \leq 2 \end{cases}$$

which has a jump discontinuity at $t = 0$. In fact, the only way to change $g(x)$ to make the g -length function continuous is to remove the middle piece of the $g(x)$ function and translate one or the other pieces right or left to again make $g(x)$ continuous. But that would have the effect of moving the two pieces of the graph of $\ell_g(I)$ together. The jump discontinuity becomes a hole in the graph, easily filled. But in this case, the changed $g(x)$ has been made piecewise monotone!

One can show that for a piecewise monotonic function φ , a distribution function for φ is

$$\Psi : \mathbb{R} \rightarrow \mathbb{R}, \quad \Psi_\varphi(t) = \ell_\varphi((-\infty, t)).$$

We can use this for:

Theorem 11. Let φ be a T -periodic function of $\mathbb{R}$ such that $\varphi_T = \varphi|_{[0, T]}$ is piecewise monotone. If $\alpha \notin \mathbb{Q}$ and $t_0 \in \mathbb{R}$, then the sequence $x_n = \varphi(t_0 + n\alpha T)$ has an asymptotic distribution with distribution function

$$\Phi_{\{x_n\}}(t) = \frac{1}{T} \Psi_\varphi(t) = \frac{\ell(\varphi^{-1}((-\infty, t)))}{T}.$$

We won't prove this or study it in any more detail. But there is an interesting conclusion to draw from this. In the theorem, the sequence of samples of the T -periodic function φ has the same distribution function as the actual function φ , (defined over the period, that is) precisely when the sampling is taken at a rate which is an irrational multiple of the period T . In this way, the sequence, over the long term, will fill out the values of φ over the period in a dense way. In a way, one can recover the function φ from a sequence of regular samples of it only if the sampling is done in a way which ultimately allows for all regions of the period to be visited evenly. This is a very interesting result.

In the book is an actual calculation of the distribution function for the sequence $\{\sin n\}$. Since the natural numbers are not a rational multiple of 2π , the period of the sine function, this distribution function

is precisely the same as that distribution function of the smooth function $f(x) = \sin x$, defined on the interval $[0, 2\pi]$. Take a good look at this example.

LINEAR FLOWS ON THE 2-TORUS

We will skip Section 4.2.2 on the distribution of first digits of powers, and proceed to Section 4.2.3 and another application of irrational circle maps. to start, however, recall what a flow is: Let $\vec{x} = f(\vec{y})$, $vecx(0) = \vec{x}_0 \in \mathbb{R}^n$ be an IVP, where the vector field $f(\vec{y}) \in C^1$. This IVP defines a flow on $\mathbb{R}^n$. For $I \subset \mathbb{R}$ an interval containing 0, define a continuous map $\varphi : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ that satisfies the following:

- $\forall T \in I$, $\varphi^t = \varphi(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism (for a given choice of t , this is simply the time- t map of the IVP).
- $\forall s, t \in I$, where $s + t \in I$, one has

$$\varphi^s \circ \varphi^t(\vec{x}) = \varphi^{s+t}(\vec{x}).$$

Now suppose that $S^1 = \{e^{2\pi ix} \in \mathbb{C}\}$, and $\frac{dx}{dt} = \alpha$, $x(0) = x_0$ is an IVP defined on S^1 . This is solved by $x(t) = \alpha t + x_0$, which can also be written in flow form $\varphi_\alpha^t(x) = \alpha t + x$. Notice in this last expression, we have included the subscript α to denote the dependence of the flow on the value of the parameter α . here the time-1 map is just

$$\varphi_\alpha^1(x) = \alpha + x = R_\alpha(x), \quad x \in S^1.$$

The time-1 map is just a rotation map of the circle by α . Keep in mind, however, that the IVP will share the same time 1 map as the new IVP given by $\frac{dx}{dt} = \alpha + 1$, $x(0) = x_0$. However, the original flows are very different! Linear flows on S^1 are not very interesting. They differ only by speed (and possibly direction), and ultimately, all look like continuous rotations of the circle, whether α is rational or not.

Next class, we will generalize a bit by increasing dimension by 1. And we will find that linear flows in this new space are again quite interesting.