

MATH 421 DYNAMICS

Week 5 Lecture 2 Notes

LINEAR MAPS OF $\mathbb{R}^2$ (CONT'D.)

We ended last time with the proposition detailing the functional form of the Fibonacci Sequence. Namely:

Proposition 3.1.11. *Given the second order recursion $b_{n+2} = b_{n+1} + b_n$ with the initial data $b_0 = b_1 = 1$, we have*

$$b_n = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu},$$

where $\lambda = \frac{1+\sqrt{5}}{2}$ and $\mu = \frac{1-\sqrt{5}}{2}$.

We showed that λ and μ were the eigenvalues of a matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and that the linear map on $\mathbb{R}^2$ given by A , $\vec{v}_{n+1} = A\vec{v}_n$, is in fact the first-order vector recursion for the second-order recursion in the proposition under the assignment $\vec{v}_n = \begin{bmatrix} b_n \\ b_{n+1} \end{bmatrix}$. This reduction-of-order technique for the study of recursions is quite similar to (and is the discrete version of) the technique of studying the solutions of a single, second-order, homogeneous, ODE with constant coefficients by instead studying the system of two first-order, linear, constant-coefficient, homogeneous ODEs. In fact, this analogy is much more robust, which we will see in a minute.

First, a couple of notes:

- For very large n ,

$$b_n = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu} \sim K\lambda^{n+1}.$$

Thus the growth rate of terms in the Fibonacci sequence are not exponential. They do, however, tend to look more and more exponential as n gets large. In fact, we can say the Fibonacci sequence displays *asymptotic exponential* growth, or that the sequence grows *asymptotically exponentially*.

- Start with the initial data $\vec{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and plot $\mathcal{O}_{\vec{v}_0}$ in the plane. What you will find is that the iterates of $\mathcal{O}_{\vec{v}}^+$ will live on two curves of motion (there will be flipping here across the $y = \lambda x$ eigenline. Why does this happen?) and tend toward the λ -eigenline as they grow off of the page (see the figure). Getting closer to the λ -eigenline means that the growth rate is getting closer to the growth rate ON the λ -eigenline. But on this line, growth is purely exponential!. With growth factor $\lambda_i 1$.
- Every other point $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is really just another set of initial data for the second-order recursion (or the first-order vector version). Start taking iterates and plot and you will see that these orbits will also live on either one or will flip between two curves of motion and the phase diagram in the figure will tell you the ultimate fate of the orbits. Doing this, you should ask yourself the following questions:
 - Q. Can you find starting data which lead to a sequence which does NOT tend to run off of the page as n goes to infinity?
 - Q. if so, can you find starting data in which both b_0 and b_1 are integers? Why or why not?

In general, let $a_{n+2} = pa_n + qa_{n+1}$ (careful of the order of the terms in this expression). Then we can construct a first-order vector recursion

$$\vec{v}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A\vec{v}_n, \text{ for } A = \begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix}.$$

The characteristic equation of A is $r^2 - qr - p = 0$, with solutions $r = \frac{q \pm \sqrt{q^2 + 4p}}{2}$.

Proposition 3.1.13. *If $\begin{bmatrix} 0 & 1 \\ p & q \end{bmatrix}$ has two distinct eigenvalues $\lambda \neq \mu$, then every solution to the second-order recursion $a_{n+2} = pa_n + qa_{n+1}$ is of the form*

$$a_n = x\lambda^n + y\mu^n$$

where $x = \alpha v_1$ and $y = \beta w_1$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ are respective eigenvectors of λ and μ , and α and β satisfy the vector equation

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \alpha\vec{v} + \beta\vec{w}$$

Remark 1. Hence the general second-order recursion and the first-order vector recursion carry the same information, and the latter provides all of the information necessary to completely understand the former. The method of solution is quickly discernable: Given a second-order recursion, calculate the data from the matrix A in the corresponding first-order vector recursion, including the eigenvalues and a pair of respective eigenvectors. Use this matrix data along with the initial data given with the original recursion to calculate the parameters in the functional expression for a_n .

Example 2. Go back to the original Fibonacci recursion $a_{n+2} = a_{n+1} + a_n$, with initial data $a_0 = a_1 = 1$.

The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ has $\lambda = \frac{1+\sqrt{5}}{2}$ and $\mu = \frac{1-\sqrt{5}}{2}$ (as before) and using the notation of Proposition

3.1.13, one can calculate representative eigenvectors as $\vec{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ \mu \end{bmatrix}$. Thus $v_1 = w_1 = 1$. To calculate α and β , we have to solve the vector equation

$$\begin{aligned} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} &= \alpha\vec{v} + \beta\vec{w} \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \alpha \begin{bmatrix} 1 \\ \lambda \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu \end{bmatrix} \end{aligned}$$

This is solved by $\alpha = \frac{1-\mu}{\lambda-\mu}$ and $\beta = \frac{\lambda-1}{\lambda-\mu}$ (verify this calculation!). Hence we have $x = \alpha v_1 = \frac{1-\mu}{\lambda-\mu}$ and $y = \beta w_1 = \frac{\lambda-1}{\lambda-\mu}$, and our formula for the n th term of the sequence is

$$a_n = \frac{(1-\mu)\lambda^n + (\lambda-1)\mu^n}{\lambda-\mu}.$$

This does not look like the form in Proposition 3.1.11, however. But consider that the term

$$(1-\mu) = \frac{2}{2} - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} = \lambda,$$

and similarly $(\lambda-1) = -\mu$, we wind up with

$$a_n = \frac{(1-\mu)\lambda^n + (\lambda-1)\mu^n}{\lambda-\mu} = \frac{\lambda \cdot \lambda^n + (-\mu) \cdot \mu^n}{\lambda-\mu} = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda-\mu},$$

and we recover Proposition 3.1.11 precisely.

Exercise 1. Perform this calculation for the second-order recursion in the Lemmings Problem, and use it to calculate the population of lemmings today, given that the initial population was given in 1980.

There is a very nice discussion in Section 3.2 of the relationship between the matrices found in first-order, 2-dimensional homogeneous linear systems of ODEs and the corresponding matrices of the discrete, time-1 maps of those systems. This is good reading, and truly exposes a fact that is commonly confusing among new students to this discipline: Namely, why is it that for a ODE system with coefficient matrix A , the **sign** of the eigenvalues determines the stability of the equilibrium solution at the origin. But for a linear map of $\mathbb{R}^n$, it is the size of the **absolute values** of the eigenvalues that determine the stability of the fixed point at the origin. The matrix of the time-1 of an ODE system is NOT the same matrix as the coefficient matrix of the system. The two matrices are certainly related, but they are not identical. Furthermore, ANY ODE system has a time-1 map. But only certain types of linear maps correspond to the time-1 maps of ODE systems. It is really all about the exponential map. Try to understand well what this last statement means. We will not develop this section in the class.

RECURRENCE AND MORE COMPLICATED BEHAVIOR

So far, we have explored many systems and contexts where dynamical systems have exhibited simple behavior, or fairly simple behavior. We will now begin to explore more complicated behavior than before. However, to start, we will stay with maps of the type you have already played with. But we will change the place on which they are acting. This, in and of itself, changes the nature of the orbits. It turns out that when the space is Euclidean, orbits can converge to something or wander away toward the edge of the space. However when a space is compact, roughly that its edges are not infinitely far away, and the edges are in fact in the space, then an orbit that does not converge to any particular thing must go somewhere within the space. How to describe where it goes will take us to behavior which is more complicated than what we have already seen. To begin, consider the definition:

Definition 3. For $f : X \rightarrow X$ a continuous map on the metric space X , a point $x \in X$ is called (*positively*) *recurrent* with respect to f if there exist a sequence of natural numbers $n_k \rightarrow \infty$ where $f^{n_k}(x) \rightarrow x$.

In the simple dynamical systems we studied so far, the only recurrent points were fixed and periodic points (this makes sense, right?). However, non-periodic points can also be recurrent. This chapter begins a study of relatively simple maps that exhibit much more complicated behavior. And this behavior is captured in this notion of recurrence.

ROTATIONS OF THE CIRCLE

Again, think of S^1 either as the set of unit modulus numbers of the complex plane

$$S^1 = \left\{ z \in \mathbb{C} \mid z = e^{2\pi i \theta}, \theta \in \mathbb{R} \right\},$$

or as the quotient space of the real line modulo the integers, $S^1 = \mathbb{R}/\mathbb{Z}$. Recall, for $x, y \in \mathbb{R}$, denote $\bar{x}, \bar{y}$ their respective points in S^1 under the exponential map $\rho : \mathbb{R} \rightarrow S^1$, $\rho(\theta) = e^{2\pi i \theta}$.

- Here $\bar{x} = \bar{y}$ iff $x - y \in \mathbb{Z}$, or $x \equiv y \pmod{1}$.
- $\bar{x}, \bar{y}$ are the equivalence classes of points in $\mathbb{R}$ under the equivalence relations imposed on $\mathbb{R}$ by the map ρ .

In this last interpretation, one can imagine S^1 to be the unit interval $[0, 1]$ in $\mathbb{R}$ where one agrees to identify the endpoints (hence the notation I sometimes use in class that $0 = 1$).

One can define a metric on S^1 by simply inheriting the one it has as it sits in $\mathbb{C}$ (or if you will, $\mathbb{R}^2$). This is essentially the Euclidean metric and measures the straight line distance in the plane between two points. Really, this is the length of the chord, or secant line, joining the points. See the figure. But also, we can define a distance between points by the arc length between them. In some ways, this is preferable, since in the abstract, S^1 doesn't really sit anywhere. There is no interior and exterior of S^1 , unless you call the actual points in the curved line making the circle the interior points. The problem with using arc length to determine the distance between points is that there are two distinct paths going from one point to another. There must be a determination as to which one to choose. Choosing the minimal path is a nice choice, but how does one do this mathematically. The answer lies within the view that S^1 really is the real line $\mathbb{R}$ infinitely coiled up like a slinky by the exponential map ρ above, and length in $\mathbb{R}$ is easy to describe, and passes through this map, at least locally:

Define

$$d(\bar{x}, \bar{y}) = \min \left\{ |x - y| \mid x, y \in \mathbb{R}, x \in \bar{x}, y \in \bar{y} \right\}.$$

The figure below shows the equivalence classes of the points $\bar{x} = \frac{1}{3}$ and $\bar{y} = \frac{3}{4}$. Choosing arbitrary representatives x and y and calculating their distance in $\mathbb{R}$ will lead to many different results. However, the minimum distance between representatives of these two classes is well-defined and in this case, $d(\bar{x}, \bar{y}) = \frac{5}{12}$. Notice that really, the closest two distance one can calculate in $\mathbb{R}$ correspond to the arc lengths in S^1 along the two paths joining $\bar{x}$ and $\bar{y}$.

Lemma 4. *These two metrics are equivalent.*

Proof. This is a really good exercise, □

Denote by R_α the rotation of S^1 by the angle α . We have parameterized S^1 as the unit interval in $\mathbb{R}$, with $0 = 1$. So even though α technically can be any real number, rotating by α and rotating by $\alpha + n$, where $n \in \mathbb{Z}$ amounts to the same thing. (Note that this would definitely not be the case for a continuous dynamical system given by $\dot{x} = \alpha x$, $x \in S^1$. Can you see why?) Here $R_\alpha(\bar{x}) = \bar{x} + \alpha$. In complex notation, we view rotations as linear maps, with multiplication by the factor $z_\alpha = e^{2\pi i \alpha}$, so that $R_\alpha(z) = z_\alpha z$. In each case, then

$$R_\alpha : S^1 \rightarrow S^1, \text{ with either } R_\alpha^n(\bar{x}) = \bar{x} + n\alpha \text{ or } R_\alpha^n(z) = z_\alpha^n z.$$

- Q.** What can we say about the dynamics of a circle rotation?
- Q.** What if $\alpha \in \mathbb{Q}$?
- Q.** What if $\alpha \notin \mathbb{Q}$?

The quick answers are that, when α is rational, all orbits are periodic, and all of the same period. When α is not rational, then there are no periodic orbits at all. I ask you to show this in the exercises, and the trick really is to understand well what R_α^n looks like for each n , and what it means for a point to be periodic in the circle. However, a deeper concern is the following: Without fixed or periodic points in S^1 for what I will call an *irrational rotation*, the question is, where do the orbits go? They cannot converge to a point in the circle, since in many case and really in general, if they converged to a point in S^1 , then that point would have to be a fixed point. The answer is that they go everywhere. And that tell one a lot about the dynamics.

Remark 5. For the parameterization of S^1 given by the interval $[0, 1)$, whether α is rational or not is really with respect to the integer 1. Suppose instead we parameterized S^1 via the interval $[0, 2\pi)$, another rather common parameterization. Then rotation half way around the circle is given by R_π , where $\alpha = \pi$ is irrational! But here every point is 2-periodic. The correct conclusion to draw here is that whether α is rational or not in $\mathbb{R}$ does not matter. What matters dynamically is whether α is rational or not with respect to the total arc-length of the circle. We will revisit this in detail in a while.

Proposition 6. *For R_α an irrational rotation of S^1 , all orbits are dense in S^1 .*

(*idea*). Really, the idea is the following:

- Show the forward orbit of any $\bar{x}$ is not periodic (you will do this in the exercises).
- Show that $\forall \epsilon > 0, \exists N \in \mathbb{N}$, such that $d(R_\alpha^N(\bar{x}), \bar{x}) < \epsilon$.
- Show that this is true for all $\bar{x}$.

□

Note: All rotations are invertible, right? Really, they are all homeomorphisms. Here $R_\alpha^{-1}(\bar{x}) = R_{-\alpha}(\bar{x})$.

So to show density, we have to show that the orbit of $\bar{x}$ will visit any open neighborhood of $\bar{x}$. Here is a nice technique for showing this:

Continued Fraction Representation. The continued fraction representation (CFR) of a real number is a representation of real numbers as a sequence of integers in a way which essentially determines the rationality of the number. This is very much like the standard decimal representations of real numbers, in that it also (our usual base-10 version is a good example) provides a ready way to represent all real numbers. However, the sequence of integers which represent a real number in a base-10 decimal expansion represent some rational numbers as finite-length sequences (think $\frac{11}{8} = 1.375$), and others as infinite length sequences (think $\frac{4}{9} = 0.44444 \dots$). The CFR instead is a base-free representation in which all and only rational number representations are the finite length sequences. Plus, the CFR is another nice way to approximate a real number by either truncating its sequence or simply not calculating the entire sequence.

Indeed, in the CFR, Any real number in $(0, 1)$ can be written as $\frac{1}{s} \in$, where $s \in (1, \infty)$. More generally, then, any real number r can be written as an integer and a real number in $(0, 1)$; as

$$r = n + \frac{1}{s}, \text{ where } n \in \mathbb{Z}, \text{ and } s \in (1, \infty).$$

If $s \in \mathbb{N}$, then this expression is considered the CFR of r (it is sometimes written then $r = [m; s]$; For example, $\frac{5}{2} = [2; 2]$).

Now suppose $s \notin \mathbb{N}$. Then since $s \in (1, \infty)$, $s = m + \frac{1}{t}$, for $m \in \mathbb{N}$, and $t \in (1, \infty)$. Thus,

$$r = n + \frac{1}{\frac{1}{m + \frac{1}{t}}}, \text{ where } n \in \mathbb{Z}, m \in \mathbb{N}, \text{ and } t \in (1, \infty).$$

Again, if $t \in \mathbb{N}$, then we stop and $r = [n; m, t]$ is the CFR of r . If it is not, we again let $t = p + \frac{1}{u}$, for $p \in \mathbb{N}$ and $u \in (1, \infty)$ so

$$r = n + \frac{1}{\frac{1}{m + \frac{1}{p + \frac{1}{u}}}}, \text{ where } n \in \mathbb{Z}, m, p \in \mathbb{N}, \text{ and } u \in (1, \infty).$$

Again, if $u \in \mathbb{N}$, we stop and the CFR of r is $[n : m, p, u]$. If not, then we continue indefinitely. The CFR is a finite sequence iff $r \in \mathbb{Q}$.

Example 7. So let R_α be a rotation of S^1 for $\alpha = \frac{1}{3 + \frac{1}{5 + \frac{1}{c}}}$, where $c > 1$, and $c \notin \mathbb{Q}$. Then it turns out that $\alpha \notin \mathbb{Q}$. Let's start the construction which would establish the middle bullet point in the above proof idea. To start, it should be obvious that $\frac{1}{4} < \alpha < \frac{1}{3}$ (why?). Denote, as in the figure, $a = R_\alpha(0)$, $b = R_\alpha^2(0)$, $c = R_\alpha^3(0)$, and $d = R_\alpha^4(0)$. One of c and d winds up being the early, closest approach to 0 of the orbit $\mathcal{O}_0^+$. But which is smaller, $\delta = 1 - 3\alpha$, or $\beta = 4\alpha - 1$? Well, the first calculation can be performed via the partial CFR given here:

$$\begin{aligned} \delta = 1 - 3\alpha &= 1 - \frac{3}{3 + \frac{1}{5 + \frac{1}{c}}} = \frac{3 + \frac{1}{5 + \frac{1}{c}} - 3}{3 + \frac{1}{5 + \frac{1}{c}}} \\ &= \frac{\frac{1}{5 + \frac{1}{c}}}{3 + \frac{1}{5 + \frac{1}{c}}} = \frac{1}{16 + \frac{3}{c}}. \end{aligned}$$

I ask you to calculate β in the exercises, and show that it is greater than δ . Hence, the third iterate is the first closest return of $\mathcal{O}_0^+$ to 0).

Q. Will the orbit ever get closer to 0?

Q. If it will, then which iterate?

These questions will be necessary to prove that the orbit will eventually get arbitrarily close to 0.

We will complete this idea next time.