MATH 421 DYNAMICS

Week 3 Lecture 2 Notes

Last class, we started the discussion on First Return Maps. Recall that one can view the first return map as a local version (only defined near interesting orbits) of the more globally defined time-t map (defined over all of phase space). We left the discussion with a 2-dimensional example: Consider the system on polar coordinates (Phase place pic to follow):

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= 1 \end{aligned} \quad . \tag{1}$$

Again, without solving this system, we can say a lot about how solutions behave:

- The system is autonomous, so when you start does not matter, and the vector field is constant over time,
- The only equilibrium solution is at the origin. The second equation in the system really states that no point is fixed when θ is uniquely defined (on $[0, 2\pi)$, that is) for a choice of point in the plane. But the origin is special in polar coordinates.
- $r(t) \equiv 1$ is a periodic solution (the only one?) and called a *cycle*. What is the period?
- $r(t) \equiv 1$ is asymptotically stable as a cycle, and is called a *limit cycle*. can you see why?

If we define

$$I = \left\{ [\alpha, \beta] \subset \text{ vertical axis } \middle| 0 < \alpha < 1, \beta > 1 \right\},$$

then for each $x \in I$, $x = r_x(0)$, the initial value for some solution $r_x(t)$ of Equation 1. One can show that in this case, every solution $r_x(t)$, where $x \in I$ will again intersect I in some positive time. Last class, we defined y_x the position in i of the first such crossing. The map $\phi : x \mapsto y_x$ defines a discrete dynamical system on I.

Some properties of this discrete dynamical system should be clear:

- The dynamics are simple on I: There is a unique fixed point at $x = 1$ corresponding to the limit cycle crossing. This fixed point is asymptotically stable so that $\forall x \in I$, $\mathcal{O}_x \longrightarrow 1$. Thus this discrete dynamical system is a contraction on I.
- The same can be said for the system

$$\begin{aligned} \dot{r} &= r\left(\tfrac{1}{2} - r\right)(r-1)\left(\tfrac{3}{2} - r\right) \\ \dot{\theta} &= -1 \end{aligned} \quad , \tag{2}$$

  but only if I is chosen more carefully: Here

$$I = \left\{ [\alpha, \beta] \subset \text{ vertical axis } \middle| \frac{1}{2} < \alpha < 1, 1 < \beta < \frac{3}{2} \right\}.$$

- In this last system, what happens near the cycles $r(t) \equiv \dfrac{1}{2}$ and $r(t) \equiv \dfrac{3}{2}$? Is there some kind of discrete dynamical system in the form of a first return map near there also?

Recall also that for any "nice" ODE in $\mathbb{R}^n$ (the definition of nice here is mathematical), in a neighborhood of an equilibrium solution, one can "linearize" the system. This means that, when possible, one can associate to this system a linear system whose equilibrium solution at the origin has the same properties as that of the original system, at least close by the equilibrium in study, Think of the tangent line approximation of a function at a point an you get the idea. Indeed, for the C^1-system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y), \end{aligned}$$

Date: February 19, 2010.

if (x_0, y_0) is an equilibrium solution, then the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is the linearized ODE system in a neighborhood of (x_0, y_0).