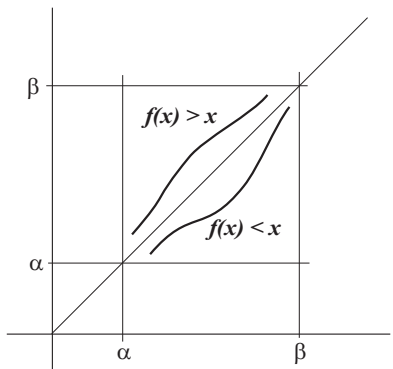


MATH 421 DYNAMICS

Week 3 Lecture 1 Notes



We begin today's lecture with a proof of our last assertion from last class:

Proposition 1. For the C^0 map $f : [\alpha, \beta] \rightarrow [\alpha, \beta]$, f must have a fixed point.

Remark 2. Again, visually, this should make sense. Try to draw the graph of a continuous function in the unit square in a way that it does NOT intersect the diagonal.

Proof. Suppose for now that f has no fixed point on (α, β) (this seems plausible, since our example above $f(x) = x^2$ on $[0, 1]$ satisfies this criterion). Then, it must be the case that for all $x \in (\alpha, \beta)$, either **(a)** $f(x) > x$, or **(b)** $f(x) < x$. This means that the entire graph of f lies above the diagonal, or below it, respectively (See the figure).

If we are in situation **(a)**, then for any choice of $x \in (\alpha, \beta)$, $\mathcal{O}_x$ will be an increasing sequence in $[\alpha, \beta]$. It is a known fact in analysis that an increasing sequence which is bounded above must converge (this is taught in our course 110.113 Honors Single Variable Calculus). For case **(b)**, the orbit is strictly decreasing, and will be bounded below since the entire sequence lives in the closed interval $[\alpha, \beta]$. Thus we can say that $\mathcal{O}_x \rightarrow x_0$, for some $x_0 \in [\alpha, \beta]$ (we can also say $\lim_{n \rightarrow \infty} f^n(x) = x_0$).

What can this point x_0 look like? Well, for starters, it must be a fixed point! To see this,

$$f(x_0) = f\left(\lim_{n \rightarrow \infty} f^n(x)\right) = \lim_{n \rightarrow \infty} f^{n+1}(x) = \lim_{n \rightarrow \infty} f^{n+1}(x) = x_0.$$

Since there are no fixed points in (α, β) by assumption, it must be the case that either $x_0 = \beta$ (case **(a)**), or $x_0 = \alpha$ (case **(b)**). $\square$

This last proof immediately tells us the following:

Proposition 3. Let the C^0 -map $f : [\alpha, \beta] \rightarrow [\alpha, \beta]$ be non-decreasing, and suppose there are no fixed points on (α, β) . Then either

- exactly one end point is fixed and $\forall x \in [\alpha, \beta]$, $\mathcal{O}_x$ converges to the fixed end point, or
- Both end points are fixed, one is an attractor and the other is a repeller.

And if in the second case above, f is also invertible (f increasing is enough for this to be the case), then for all $x \in (\alpha, \beta)$, $\mathcal{O}_x$ is forward asymptotic to one end point, and backward asymptotic to the other.

For the first case, think of the map $f(x) = \sqrt{x}$ on the closed interval $[1, \infty)$. For the second case, think $g(x) = x^2$ on $[0, 1]$. We can make the above proposition a bit clearer by adding to our definitions of orbits:

Definition 4. Let $f : [\alpha, \beta] \rightarrow [\alpha, \beta]$ be C^0 and invertible (it has continuous inverse $f^{-1}(x)$). Then

- $\mathcal{O}_x := \left\{ y \in [\alpha, \beta] \mid y = f^n(x), n \in \mathbb{Z} \right\}$
- $\mathcal{O}_x^+ := \left\{ y \in [\alpha, \beta] \mid y = f^n(x), n \in \mathbb{N} \right\}$

$$\bullet \mathcal{O}_x^- := \left\{ y \in [\alpha, \beta] \mid y = f^n(x), \ n \in -\mathbb{N} \right\}.$$

Then in the last statement of the Proposition above, we can say: For f invertible, then $\forall x \in (\alpha, \beta)$, either $\mathcal{O}_x^+ \longrightarrow \alpha$ and $\mathcal{O}_x^- \longrightarrow \beta$, or $\mathcal{O}_x^+ \longrightarrow \beta$ and $\mathcal{O}_x^- \longrightarrow \alpha$.

Let's elaborate on this forward/backward orbit thing. Suppose now that $f : X \rightarrow X$ is a C^0 -map on some subset of $\mathbb{R}^n$, and suppose $\exists x \in X$, where

$$\mathcal{O}_x^- \longrightarrow a \text{ and } \mathcal{O}_x^+ \longrightarrow b.$$

Definition 5. x is said to be heteroclinic to a and b if $a \neq b$, and homoclinic to a if $a = b$.

You have certainly seen heteroclinic and homoclinic orbits before. Think of the phase plane of the undamped pendulum. It is the famous picture below (to follow). Here the separatrices are heteroclinic orbits from the unstable equilibrium solution at $(2n\pi, 0)$, $n \in \mathbb{Z}$, and $(2(n \pm 1)\pi, 0)$. Although, in reality, there is a much more accurate picture of the phase space of the undamped pendulum. The vertical variable (representing the instantaneous velocity of the pendulum ball, actually) takes values in $\mathbb{R}$, while the horizontal variable (representing the angular position of the pendulum ball with respect to downward vertical position) is in reality 2π -periodic. Truly, it takes values in the circle S^1 :

$$S^1 = \text{unit circle in } \mathbb{R}^2 = \left\{ e^{i\theta} \in \mathbb{C} \mid \theta \in [0, 1) \right\}.$$

Thus, the phase space is really a cylinder, and really only has two equilibrium solutions; one at $(0, 0)$, and the other at $(\pi, 0)$. In this view, which we will elaborate on later, there are only two separatrices, and both are homoclinic to the unstable equilibrium at $(\pi, 0)$. Also, it becomes clear once the picture is understood that ALL orbits of the undamped pendulum, except for the separatrices, are periodic. However, the period of these orbits is certainly not all the same. And there is NO bound to how long a period may actually be. See if you can fully grasp this.

Other things to think about:

- There cannot exist homoclinic points for f a nondecreasing map on an interval. Why not?
- Is it possible to have a homoclinic fixed point on S^1 ? (Hint: Construct one. There is an example of an interval map that we already studied in this class that can be modified to work here.)

Forcing a map of an interval to be nondecreasing and forcing the interval to be closed really restricts the types of dynamics that can happen. For instance,

Proposition 6. *Let $f : [\alpha, \beta] \rightarrow [\alpha, \beta]$ be C^0 and nondecreasing. Then $\forall x \in [\alpha, \beta]$, either x is fixed, or asymptotic to a fixed point. And if f is increasing (and thus invertible), then $\forall x \in [\alpha, \beta]$, either x is fixed or heteroclinic to adjacent fixed points.*

Clearly, the dynamics, although more complicated than for contraction maps, are nonetheless rather simple for nondecreasing interval maps (and even more so for invertible interval maps. Thus goes the second stop in our exploration of dynamical systems from simple to complex.

Before moving on, one more statement. The last Proposition presents another interesting phenomenon:

- In the case of an increasing (invertible) map, where the fixed points are separated by regions of non-fixed points (remember that for the map $f(x) = x$, all points are fixed), all orbits that start

between two fixed points wind up staying within the same two fixed points. This lends weight to the Proposition above about the interval without fixed points in the interior and what happens at the endpoints. Thus maps interval maps can be broken down to maps on the subintervals formed by each pair of adjacent fixed points.

- If every point is either fixed or tends to a fixed point for an non-decreasing map, then how possibly can a period-2 point occur? periodic orbit are neither fixed nor do they “tend” anywhere but themselves. hence another interesting property of nondecreasing interval maps: There cannot exist non-trivial periodic points. We will see later just how special this is.
- If we vary an increasing interval map slightly, usually the dynamical behavior of the map stays the same (the number and type of fixed points does not change, even though their position may a bit). This may not be the case for a non-decreasing map: A slightly perturbed increasing map will remain increasing, while a slightly perturbed map with a flat interval may not remain nondecreasing. Think about that.
- Sometimes, a small change in an increasing map may lead to a big change in the number and type of fixed points (i/e/, a big dynamical change!). Consider the three graphs below. Do you recognize this behavior? Have you ever seen a bifurcation in a mechanical system with a parameter?

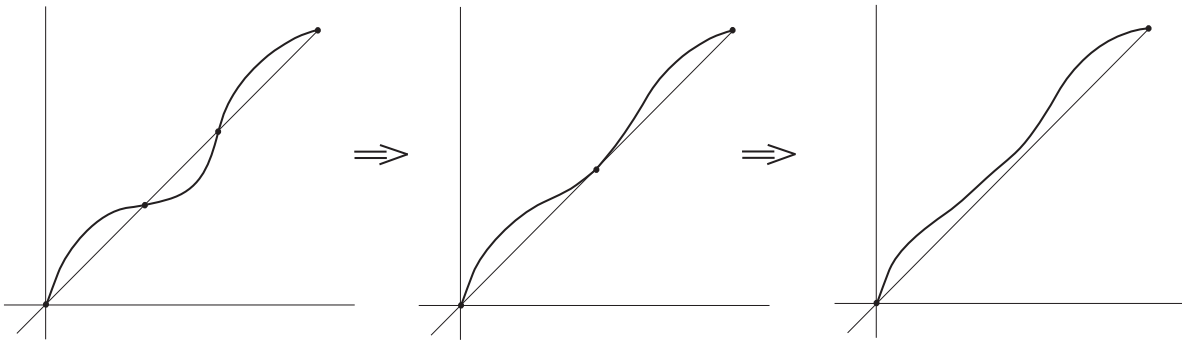


FIGURE 1. A bifurcation in an interval map.

1. TIME- t AND FIRST RETURN MAPS

We will skip the sections on phase lines and 1-dimensional ODEs due to their full coverage in the prerequisite course for this one: 110.302 Differential Equations. Instead, we will jump directly to Section 2.4.3 in the book.

Recall that the time-1 map of an ordinary differential equation defines a discrete dynamical system on the phase space. Indeed, for $\mathbf{x} \in \mathbb{R}^n$, the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ defines the map

$$\phi_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \phi_1 : \mathbf{x}(0) \mapsto \mathbf{x}(1)$$

which is a transformation of $\mathbb{R}^n$. Really, $t = 1$ is only one such example, and any t will work, so long as the system solutions are defined (and unique, for the most part). Place picture of phase space time- t map here.

There is another kind of discrete dynamical system that comes from a continuous one: the First Return Map. One can view the first return map as a local version (only defined near interesting orbits) of the more globally defined time- t map (defined over all of phase space). Let's start with a 2-dimensional version.

Consider the system on polar coordinates (Phase plane pic to follow):

$$(1) \quad \begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= 1 \end{aligned}.$$

Without solving this system (although this is not difficult as the equations are uncoupled), we can say a lot about how solutions behave:

- The system is autonomous, so when you start does not matter, and the vector field is constant over time,
- The only equilibrium solution is at the origin. The second equation in the system really states that no point is fixed when θ is uniquely defined (on $[0, 2\pi)$, that is) for a choice of point in the plane. But the origin is special in polar coordinates.
- $r(t) \equiv 1$ is a periodic solution (the only one?) and called a *cycle*. What is the period?
- $r(t) \equiv 1$ is asymptotically stable as a cycle, and is called a *limit cycle*. can you see why?

Now define

$$I = \left\{ [\alpha, \beta] \subset \text{vertical axis} \mid 0 < \alpha < 1, \beta > 1 \right\}.$$

or each $x \in I$, $x = r_x(0)$, the initial value for some solution $r_x(t)$ of Equation 1. Let y_x be the point in I which corresponds to the earliest positive time that the resulting $r_x(t)$ again crosses I . One, it must cross again (why?), and two, really $y_x = r_x(2\pi)$. Then the map $\phi : x \mapsto y_x$ defines a discrete dynamical system on I .

Next class, we will look at some of the properties of this discrete dynamical system.