MATH 421 DYNAMICS

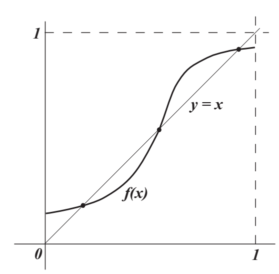
Week 2 Lecture 2 Notes

The Contraction Principle proved last class is a facet of some dynamical systems which display what is called "simple dynamics": With very little information about the system (map or ODE), one can say just about everything there is to say about the system. Another way to put this is to say, that in a contraction, all orbits do exactly the same thing. Which is, they all converge to the same fixed point (equilibrium solution in the case of a continuous dynamical system.

We can build on this idea by now beginning a study of a relatively simple family of discrete dynamical systems that display slightly more complicated behavior.

1. Interval Maps

Let $f : I \to I$ be continuous map, where $I = [0, 1]$ (we will say f is a C^0-map on I, or $f \in C^0(I, I)$). The graph of f sits inside the unit circle $[0, 1]^2 = [0, 1] \times [0, 1] \subset \mathbb{R}^2$.

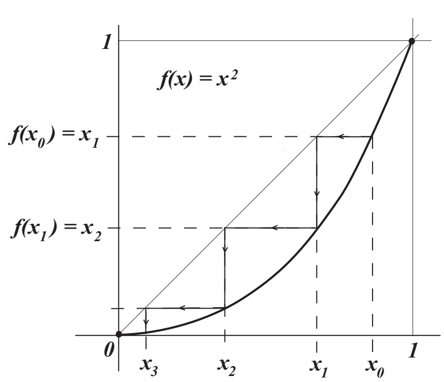


This graph intersects the line $y = x$ at precisely the points where $y = f(x) = x$, or the fixed points of the dynamical system given by f on I. Recall that the dynamical system is formed by iterating f on I, and

$$\mathcal{O}_{x_0} = \left\{ x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \dots \right\}.$$

One can track $\mathcal{O}_{x_0}$ in I (and in $[0, 1]^2$) visually via the notion of *cobwebbing*. We will use the example of $f(x) = x^2$ on I to illustrate:

Choose a starting value $x_0 \in I$. Under f, the next term in the orbit is $x_1 = f(x_0)$. Vertically, it is the height of the graph of f over x_0. Making it the new input value to f means finding its corresponding place on the horizontal axis. This is easy to see visually. The vertical line $x = x_0$ cross the graph of f at the height $x_1 = f(x_0)$. The horizontal line $y = x_1 = f(x_0$ crosses the diagonal $y = x$ precisely at the point (x_1, x_1). The vertical line through this point will again intersect the graph of f at one point (why only one?). That point will be at the height $x_2 = f(x_1)$ and constitutes the second value of the sequence $\mathcal{O}_{x_0}$. Continue zig-zagging this way and the orbit of the point x_0 visually appears. There is a nice Java applet under the title "Emergence of Chaos" on the website www.cut-the-knot.org (you will have to sort through a lot of other very interesting Java applets) which uses cobwebs to display orbits of a certain kind of interval map. Check it out.



Specific to our example $f(x) = x^2$ on I, we have two fixed points: $x = 0$ and $x = 1$. And if x_0 is chosen to be strictly less than 1, then we can easily conclude via the cobweb that $\mathcal{O}_{x_0} \longrightarrow 0$. Visually, it makes sense. Analytically, it is also intuitive; squaring a number between 0 and 1 always makes it smaller. However, can you *prove* that every orbit goes to 0 except for the orbit? We will do something like this shortly.

First we will need some definitions which will allow us to talk about the nature of fixed points in terms of what happens around them. This language is a lot like the way we classified equilibrium solutions in the ODEs class. For the moment, think of X as an interval in $\mathbb{R}$. However, in these definitions, you can readily allow X to be ANY metric space, and the absolute-value signs are simply metric distances.

Date: February 19, 2010.

Definition 1. Let x_0 be a fixed point of the C^0-map $f : X \to X$. Then x_0 is said to be

- *Poisson stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in I$, if $|x - x_0| < \delta$, then $\forall N \in \mathbb{N}$ $|f^n(x) - x_0| < \epsilon$.
- *asymptotically stable*, or an *attractor* if $\exists \epsilon > 0$ such that $\forall x \in I$, if $|x - x_0| < \epsilon$, then $\mathcal{O}_x \longrightarrow x_0$.
- a *repeller*, if $\exists \epsilon > 0$ such that $\forall x \in I$, if $0 < |x - x_0| < \epsilon$, then $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|f^n(x) - x_0| > \epsilon$.

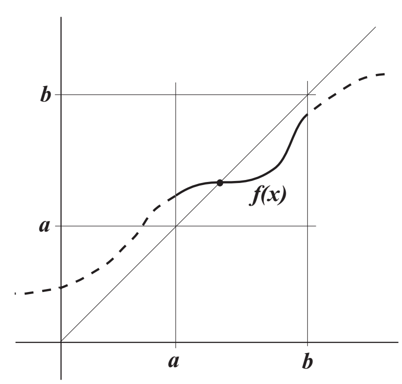
Remark 2. Asymptotically stable basically means that there is a neighborhood of the fixed point where f restricted to that neighborhood is a contraction with x_0 as the sole fixed point. Poisson stable means that given any small neighborhood of the fixed point, I can choose a smaller neighborhood where if I start in the smaller neighborhood, the forward orbit never leaves the larger neighborhood. Asymptotically stable points are always Poisson stable, but not necessarily vice versa. And a fixed point is a repeller if in a small neighborhood of the fixed point, all points that are not the fixed point itself have forward orbit that leave the neighborhood and never return.

Remark 3. In the classification of 2×2, first-order, homogeneous, linear ODE systems, one can classify the type of the equilibrium solution at the origin via a knowledge of the eigenvalues of the coefficient matrix. IN this classification, the sink (negative eigenvalues) was the asymptotically stable equilibrium, the source was the repeller, and the center (recall where the two eigenvalues were purely imaginary complex conjugates) was the Poisson stable equilibrium,

Remark 4. Back to the example of $f(x) = x^2$ on $[0, 1]$. This dynamical system has two fixed points. One can see visually (via cobwebbing or via an analysis of the properties of the orbits) that $x = 0$ is asymptotically stable. Whereas $x = 1$ is unstable and a repeller. Can you show that these two points satisfy the respective definitions? Can you analytically prove that one is an attractor and the other is a repeller?

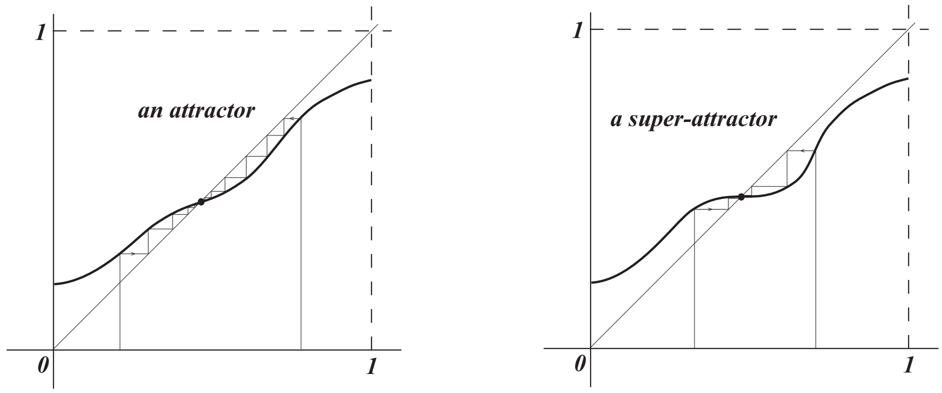
In this class, we will spend a fair amount of time on the maps $f : [0, 1] \to [0, 1]$. There are basically two reasons for this: 1) They have applications beyond simple interval maps, and 2) maps of the unit interval are really all one need study when studying intervals. To see the second point, let $f : \mathbb{R} \to \mathbb{R}$, but suppose that there exists a closed interval $[a, b]$, $b > a$ (a single point is considered a closed interval, so the condition that $b > a$ means that I want something with an interior), where

$$f\Big|_{[a,b]} : [a, b] \to [a, b].$$



- Dynamically speaking, what happens to $f(x)$ under iteration on $[a, b]$ is no different from what happens to $g(y)$ on $[0, 1]$ under the linear transformation of coordinates $y = \frac{x-a}{b-a}$.
- It is usually an $f(x)$ from above that appears in applications, and mathematically we usually only study maps like $g(y)$. For example, let's go back to the Newton Raphson Method for root location.

Definition 5. A fixed point x_0 for $f : I \to I$ where $f \in C^1$ is called superattracting if $f'(x_0) = 0$. (Why? See the picture.)



Proposition 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^3 with a root r. If $\exists \delta > 0$, $M > 0$ such that $|f'(x)| > \delta$ and $|f''(x)| < M$ on a neighborhood of r, then r is a superattracting fixed point of*

$$F(x) = x - \frac{f(x)}{f'(x)}.$$

Proof. As we have already calculated, $F'(r) = \dfrac{f(r)f''(r)}{\left[f'(r)\right]^2} = 0.$ □

We can go further, and I will state this part without proof: Since f is C^3, then F is C^1. Calculating $F'(x)$ and knowing that it is both continuous and 0 at $x = r$, there will be a small, closed interval $[a, b]$, $b > a$ with r in the interior, where $|F'(x)| < 1$. One can show that that restricted to this interval,

$$F\Big|_{[a,b]} : [a, b] \to [a, b]$$

is a λ-contraction, with a superattracting fixed point at r. In this case, all orbits of F converge exponentially to r by λ^2, even thought F is simply a λ-contraction.

Interval maps are quite general, and display tone of diverse and interesting behavior. To begin exploring this behavior, we will need to specify some types of interval maps. The first type designation is as follows:

Definition 7. A map $f : [\alpha, \beta] \to [\alpha, \beta]$ be C^0. We say f is

- *increasing* if for $x > y$, we have $f(x) > f(y)$,
- *nondecreasing* if for $x > y$, we have $f(x) \geq f(y)$,
- *non-increasing* if for $x > y$, we have $f(x) \leq f(y)$,
- *decreasing* if for $x > y$, we have $f(x) < f(y)$.

It is easy and intuitive to see how these definitions work. You should draw some examples to differentiate these types. You should also work to understand how these different types affect the dynamics a lot. For example, increasing and non-decreasing map can have many fixed points (actually, the map $f(x) = x$ has ALL points fixed!). While all non-increasing maps (hence all decreasing maps also) can have only one fixed point each. Further, increasing maps cannot have points of period two (why not?), while there exist a decreasing map with ALL points of period two (can you find it?). We will explore these in time. For now, we will start with a fact shared by ALL interval maps:

Proposition 8. *For the C^0 map $f : [\alpha, \beta] \to [\alpha, \beta]$, f must have a fixed point.*

Remark 9. Visually, this should make sense. Try to draw the graph of a continuous function in the unit square in a way that is does NOT intersect the diagonal. When you get tired of trying, read on.

We will prove this next class....