MATH 421 DYNAMICS

Week 12 Lecture 2 Notes

1. Quadratic Maps (revisited)

We begin today by going back to quadratic maps:

Let $I = [0, 1]$ and $f_\lambda : I \to I$, $f_\lambda(x) = \lambda x(1 - x)$, but this time let $\lambda \in [3, 4]$.

Definition 1. Let $x \in X$ be fixed for the map $f : X \to X$. The *basin of attraction* of x is

$$B(x) = \left\{ y \in X \middle| \mathcal{O}_y \to x \right\}.$$

- Sometimes the basin of attraction is easy to describe:

  Example 2. Let $\dot{r} = r(r - 1)$, $\dot{\theta} = 1$ be the planar ODE system. It should be obvious now that the only equilibrium solution is at the origin of the plane, and the only other "interesting" behavior is the unstable limit cycle given by the equation $r(t) \equiv 1$. Since solution are unique on all of $\mathbb{R}^2$ (and hence cannot cross), what starts inside the unit circle stays inside. And since the limit cycle is repelling, and there are no other limit cycles or equilibria inside the unit circle, it must be the case that the origin is attracting (you can also see this directly by noting that $\dot{r} < 0$, $\forall r \in (0, 1)$). hence the basin of attraction of the origin is the open unit disk

  $$B\left((0, 0)\right) = \left\{ (r, \theta \in \mathbb{R}^2 \middle| r < 1 \right\}.$$

- Sometimes, it is not:

  Example 3. Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 + c$, for $c \in \mathbb{C}$ a constant. For $c = 0$, we get a rather plain model. $\mathcal{O}_z \to 0$ $\forall |(|z) < 1$, and $\mathcal{O}_z \to \infty$ $\forall |(|z) > 1$. Do you recognize the map on the unit circle $|(|z) = 1$? It is the expanding (and chaotic) map $E_2 : S^1 \to S^1$ from before.

  Definition 4. For $P : \mathbb{C} \to \mathbb{C}$ a polynomial map, the *Julia Set* is the closure of the set of repelling periodic points of P.

  Keep this in mind. For the map E_2 in the circle, recall that the periodic points are dense in S^1 (this was a feature of chaos). And since the map is expanding, you can show that all of these periodic points are actually repelling (simultaneously!). The resulting mess is actually what a "sensitive dependence on initial conditions" is all about. Here again, the origin in $\mathbb{C}$ is an attracting fixed point, and its basin of attraction is everything inside the unit circle.

  Now, though, let c be small and non-zero. There will still be two fixed points, right? (think of solving the equation $z = f(z) = z^2 + c$. The solutions will be $z = \frac{1 \pm \sqrt{1-4c}}{2}$. For $z \in \mathbb{C}$, this always has two solutions!) The one near the origin will still be attracting, while the one near the unit circle will still be a part of a set of repelling periodic points whose closure will form a (typically) fractal structure. This is again the Julia Set for this value of c, and can be highly bizarre looking. I showed you a few examples in class.

  In sum, for general $c \in \mathbb{C}$, the Julia set is not a smooth curve. For example, let $c < -2$ be real. Then $f_c(z) = z^2 + c$ is topologically conjugate to a map of the form $x \mapsto \lambda x(1 - x)$ for $\lambda > 4$ (this conjugacy is really just a change of variables. Can you find it?) The ramifications of this being that 1) the dynamics outside of the Julia Set are rather simple (think that outside of those interesting orbits of f_λ that stay within I forever, all orbits basically go to $-\infty$). But this implies that that Julia Set is conjugate to a Cantor Set. But this also means that the Cantor Set of points whose orbits stay within I under f_λ, $\lambda > 4$, consists of the closure of a set of repelling periodic points.

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Definition 5. An m-periodic point p is called *attracting* under a continuous map f if $\exists \epsilon > 0$ such that $\forall x \in X$, where $d(x.p) < \epsilon$, then $d\left(f^n(x), f^n(p)\right) \overset{n\to\infty}{\longrightarrow} 0$.

Exercise 1. Show that for an attracting m-periodic point p, each distinct point it its orbit is also attracting.

Call the basin of attraction for an m-periodic point p the union of the basins of attraction for each point of $\mathcal{O}_p$. That is, for $\mathcal{O}_p = \left\{p.f(p), f^2(p), \ldots, f^{m-1}(p)\right\}$, the basin of attraction of p is

$$B(p) = \left\{ x \in X \middle| d(f^n(x), f^{n+k}(p)) \overset{n\to\infty}{\longrightarrow} 0 \text{ for some } k \in \mathbb{N} \right\}.$$

Proposition 6. *Let $f : [a, b] \to [a, b]$ be C^2 and concave, where $f(a) = f(b) = a$. Then f has at most one attracting periodic orbit.*

We will not prove this. However, it is useful.

Example 7. As in the figure, in all of our examples of $f_\lambda : I \to I$, where $\lambda \in [0, 3]$ there was always an attracting fixed point. However, for $\lambda = 3.1$, for example, the fixed point at $x = 0$ is repelling, and there is an attracting period-2 orbit (can you find the numeric values for this orbit?)

Theorem 8. *If f_λ has an attracting periodic orbit, then the set outside of the basin of attraction (called the universal repeller) is a nowhere-dense null set.*

Some notes:

- A nowhere dense null set in a metric space is a set that can be covered by balls whose total volume is less than ϵ.
- What can lie within the universal repeller? First, any repelling fixed or periodic points, of course. But since the logistic map is a two-to-one map, the pre-image of a fixed point consists of two points, and includes a point that was not previously fixed.

  Example 9. Let $\lambda = 3.1 = \frac{31}{10}$. It can be shown that $f_{3.1}$ has an attracting period-2 orbit. And $x_\lambda = 1 - \frac{1}{\lambda} = 1 - \frac{1}{\frac{31}{10}} = \frac{21}{31}$ is fixed under $f_{3.1}$ and repelling (check this!) But the point $1 - x_\lambda = \frac{10}{31}$ also maps to $\frac{21}{31}$. In fact, the point $1 - x_\lambda$ is always the pre-image of the fixed point x_λ due to where it sits on the graph of f_λ. Both of these points are NOT in the basic of attraction of any periodic orbit. But also, $1 - x_\lambda$ is NOT a periodic point. It is an eventually fixed point, but that is different. Now the point $1 - x_\lambda$ also has two pre-images (find them), and these two pre-images also have two pre-images. In fact, there are a countable number of pre-images that eventually get mapped onto x_λ. All of this set lies outside of the basic of attraction of any attracting periodic orbit, when x_λ is repelling. This gives you an idea of what is considered part of the universal repeller. Now think about how this set of pre-images of x_λ sit inside the interval! If you think about it correctly, you start to see just hoe fractals are born.

Example 10. For $\lambda \in \left[3, 1 + \sqrt{6}\right]$, there exists an attracting, period-2 orbit. The basin of attraction is everything except for the points 0, and $x_\lambda = 1 - \frac{1}{\lambda}$ and ALL of their pre-images.

Let's work out the situation: For $\lambda \in (2, 3)$, 0 is a repelling fixed point, x_λ is an attracting fixed point, and there are no other periodic points. In contrast, for $\lambda \in \left(3, 1 + \sqrt{6}\right)$, both $x = 0$, and x_λ are repelling fixed point, and there now exist an attracting period-2 orbit. This means that we have reached a bifurcation value for λ at $\lambda = 3$. This type of bifurcation is called a *period-doubling bifurcation*, and is visually a "pitchfork" bifurcation for the map f_λ^2. See the figure. Analytically, what happens is that the value of $|f'(x_\lambda)| < 1$ for

$\lambda < 3$ and $|f'(x_\lambda)| > 1$ for $\lambda > 3$. But these derivatives are negative, right? for the map f_λ^2, this means that the same thing happens but the derivative are positive!. Geometrically, this determines how the graph of f_λ^2 crosses the line $y = x$, and the crossing changes from over/under to under/over as we pass through the value $\lambda = 3$. And when the graph of f_λ^2 passes to the under/over configuration, it creates two new fixed points (for the f_λ^2 map). You do not see these new crossings in the original map f_λ because they are only period two points. You can cobweb them to see that they are there, though. The under/over crossing means that the derivative is greater than 1, and hence the map is expending near the fixed point (repelling). In contrast, the two new fixed points are over/under crossings, with a derivative less than 1, and hence are attracting. Again, see the figure. it is all in there.

Finally, what happens when $\lambda = 1+\sqrt{6}$. Basically, the same thing, except that the period-2 orbit becomes a repelling orbit and a period-4 orbit is born and is attracting! Another period-doubling bifurcation.

Theorem 11. *there exists a monotonic sequence of parameter values*

$$lambda_1 = 3, \quad \lambda_2 = 1 + \sqrt{6}, \quad \lambda_3 = \ldots, \quad \textit{such that } \forall \lambda \in (\lambda_n, \lambda n + 1)\,,$$

the quadratic map $f_\lambda(x) = \lambda x(1-x)$ has an attracting period-2^n orbit, two repelling fixed points at $x = 0, x_\lambda$, and one repelling period-2^k orbit for each $k = 1, \ldots, n-1$.

Notes:

- This is called a *period=doubling cascade.*
- At every new λ_n, the previous attracting periodic orbit becomes repelling, and adds (with all of its pre-images) to the universal repeller.
- The length of the intervals $(\lambda_n, \lambda n + 1)$ decrease exponentially as n increases, and go to 0 somewhere before $\lambda = 4$.
- In fact, one can calculate the exponential decay of these interval lengths:

  $$\delta = \lim_{n\to\infty} \frac{\text{length}\,(\lambda_{n-1}, \lambda_n)}{\text{length}\,(\lambda_n, \lambda_{n+1})} = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \cong 4.6992016010\cdots\,.$$

  This number has a universal quality to it, as it is always the exponential decay rate of the lengths between bifurcation values in period-doubling cascades. It is called the Feigenbaum Number.
- The full bifurcation diagram looks like the figure. At the bak edge of the cascade is a place called the transition to chaos. At this point, there are a countable number of repelling periodic points. This collection along with all of their pre-images wind up being dense in the interval, and hence cause a sensitive dependence on initial conditions, commonly found in chaotic systems. This is the Julia set, which in a chaotic system is the entire set.
- Note the self-similar structure of the bifurcation diagram. it is not a fractal, really, but it is related to many of them in interesting ways.
- Look carefully at the bifurcation diagram. Even after the transition to chaos, there seem to be regions of calm periodic behavior. These are not artifacts. In fact, there exists an attracting period-3 orbit 9can you see it?) for a small band of values of λ. This attracting period-3 orbit eventually becomes a repeller, and starts another period doubling cascade (period-6 to period-12, etc.). In fact, there exists a period doubling cascade within this diagram for each prime number n. Look carefully and check in the book in chapter 11 for more details on this fascinatingly simple complicated map.

Next class, we will finish with a short discussion of some period-doubling cascades and chaotic maps that show up in my research. These maps are polynomial automorphisms of $\mathbb{R}^3$, and are rich with unexplored behavior.