MATH 421 DYNAMICS

Week 12 Lecture 2 Notes

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1. Topological Entropy (cont'd.)

We begin today with a definition:

Definition 1. Let $h_d(f,r) := \overline{\lim}_{n\to\infty} \dfrac{\log S_d(r,n,f)}{n}$. Then
$$h_d(f) := \lim_{r\to 0} h_d(f,r)$$
is called the *topological entropy* of the map f on X.

There are many things to say about this. To start:

- Another common notation for topological entropy is $h_{top}(f)$ or $h_T(f)$ or even $h(f)$. These are, in a sense, more accurate since it turns out that the topological entropy of a map does not actually depend on the metric d, at least up to equivalence, chosen for use in its definition. This was a mistake I made in class, mentioning that even inequivalent metrics lead to the same entropy. We will use the notation $h(f)$ in our subsequent discussion.
- Contractions and isometries have no entropy:

  Proposition 2. *Let f be either a contraction or an isometry. Then $h(f) = 0$.*

  Proof. In the case of f an isometry, for any $n \in \mathbb{N}$, $d_n^f = d$, since distances between iterates of a map are the same as the original distances between the initial points. Hence the r-capacity $S_{(X,d)}(r,n,f) = S_{(X,d)}(r,f)$ does not depend on n, and hence $h(r,f) = 0$. For a contraction, the iterates of two distinct points are always closer together than the original points. Hence also here $d_n^f = d$. This leads to the same conclusion. □

- Topological entropy is a dynamical invariant (invariant under conjugacy). This means that if f is (semi-)conjugate to g, then $h(f) = h(g)$. However, it is also useful to use the contrapositive: If one has two maps where $h(f) \neq h(g)$, then it is not possible that f is (semi-)conjugate to g.
- topological entropy measures, in a way, the exponential growth rate of the number of trajectories that are r-separable after n iterations. Suppose this number is proportional to e^{nh}. Then h would be the growth rate for a fixed r, and as $r \to 0$, this h would tend to the entropy.
- defining the topological entropy for a flow is simply a matter of replacing the $n \in \mathbb{N}$ with $t \in \mathbb{R}$ in all of the definitions for the invariant. we can relate the two in a way: The topological entropy of a flow is equal to the topological entropy of its time-1 map (really, its time-t for any choice of t, since the flow provides the conjugacy of any t-map with any other).
- In practice, topological entropy is quite hard to calculate. However, in many cases, and in response to the last bullet point, the entropy is directly related to the largest Lyapunov exponent of the system, at least for C^1 systems.

Proposition 3. *For the expanding map $E_m : S^1 \to S^1$, where $E_m(x) = mx \mod 1$, and $|m| \geq 1$, $h(E_m) = \log |m|$.*

Proposition 4. *For $f : \mathbb{T}^2 \to \mathbb{T}^2$, given by $\vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$ (this was the map F_L from before), $h(f) = \log \frac{3+\sqrt{5}}{2}$.*

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Note: In both of these cases, the topological entropy of the map IS the maximum positive Lyapunov exponent of the system.

Example 5. Show that $h(E_2) = \log 2$.

To do this calculation, we will need to quantify the r-capacity of S^1 under this map. This amounts to calculating $S_{(S^1,d)}(r,n,E_2)$ for a fixed r and as a function of the iterate number n. hence we start with a good idea of what constitutes the actual size of an r-ball $B_r(x,n,E_2)$ for a choice of n. Note first that by its definition, $B_r(x,n,E_2)$ is the set of points whose distance away from x is less than r after n iterates of E_2. As the map is expanding by a factor of 2 (locally), distances double after each iterate (see the figure). Hence we will have to get closer to x when we start iterating to remain within r as we iterate. Hence $B_r(x,n,E_2)$ will shrink in size as n increases. How will it shrink?

Suppose for a minute that $r = \frac{1}{4}$. Choose an $x \in S^1$, and recall that

$$B_{\frac{1}{4}}(x,0,E_2) = \left\{ y \in S^1 \,\middle|\, d_0^{E_2}(x,y) = d(x,y) = |x-y| < \frac{1}{4} \right\}.$$

The radius of $B_r(x,n,E_2)$ is $\frac{1}{4}$ here. After one iterate, however,

$$B_{\frac{1}{4}}(x,1,E_2) = \left\{ y \in S^1 \,\middle|\, d_1^{E_2}(x,y) = \max\{|x-y|, |2x-2y|\} < \frac{1}{4} \right\}.$$

Here, it is obvious that the condition that $d_1^{E_2}(x,y) = |2x-2y| = 2|x-y| < \frac{1}{4}$ means that the actual distance between x and y would have to be $|x-y| < \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$. Hence the radius of $B_{\frac{1}{4}}(x,1,E_2)$ is only $\frac{1}{8}$. Similarly, the radius of $B_{\frac{1}{4}}(x,2,E_2)$ is only $\frac{1}{16}$, and in general we have that

$$\text{radius}\left(B_{\frac{1}{4}}(x,n,E_2)\right) = \frac{1}{4} \cdot \frac{1}{2^n}.$$

But, really, the initial size of r does not determine the relative sizes of the r-balls with respect to each other. Hence, we can say that, for any choice of $r > 0$, we have

$$\text{radius}\left(B_r(x,n,E_2)\right) = r \cdot \frac{1}{2^n}.$$

Recall that the r-capacity, $S_{(S^1,d)}(r,n,E_2)$ is the minimum number of the r-balls $B_r(x,n,E_2)$ it takes to cover S^1. Think of S^1 as being parameterized by the unit interval $[0,1]$ with the identification of 0 and 1. Then we really only need to find out how many r-balls we need for a given iterate n to cover an interval of length 1. Call this number K_n. Hence, we solve the equation (really, it is an inequality, but since adding one more ball to each quantity will not change the limit, this is an okay simplification)

$$\#\left(B_r(x,n,E_2)\right) \cdot 2 \cdot \text{radius}\left(B_r(x,n,E_2)\right) = K_n \cdot 2 \cdot r \cdot \frac{1}{2^n} = 1.$$

Which is solved by $K_n = \frac{1}{r} \cdot 2^{n-1}$. This is $S_{(S^1,d)}(r,n,E_2)$.

We now calculate

$$\begin{aligned} h\left(E_2, r\right) &= \overline{\lim_{n\to\infty}} \frac{\log S_{(S^1,d)}\left(r, n, E_2\right)}{n} \\ &= \lim_{n\to\infty} \frac{\log \frac{1}{r} \cdot 2^{n-1}}{n} \\ &= \lim_{n\to\infty} \left(\frac{\log \frac{1}{r}}{n} + \frac{\log 2^{n-1}}{n} \right) \\ &= 0 + \log 2 \cdot \left(\lim_{n\to\infty} \frac{n-1}{n} \right) = \log 2. \end{aligned}$$

Here again, the r-topological entropy does not depend on r at all, so that

$$h\left(E_2\right) = \lim_{r\to 0} h\left(E_2, r\right) = \lim_{r\to 0} \log 2 = \log 2.$$