MATH 421 DYNAMICS

Week 12 Lecture 1 Notes

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1. Box Dimension

We begin today with a definition:

Definition 1. A metric space X is called *totally bounded* if $\forall r > 0$, X can be covered by a finite set of r-balls all of whose centers are in X.

Really, this definition is technical, and is meant to account for the general metric space aspect of this discussion. That the centers need to be within X really only is a factor when the Metric space X is a subspace of another space Y (otherwise there is no "outside" of X. And in Euclidean space, the notion of totally bounded is just the common notion of bounded that you are used to.

Definition 2. For X totally bounded,

$$\mathrm{bdim}(X) := \lim_{r \to 0} \frac{-\log S_{(X,d)}(r)}{\log r}$$

is called the*box dimension* of X.

Notes:

- This concept is also called the Minkowski-Bouligard dimension, or the entropy dimension or the Kolmogorov dimension.
- This is an example of where the idea of fractional dimension comes from; some sets may look bigger then 0-dimensional, yet smaller than 1-dimensional, for example.
- In the case where this limit may not exist (I cannot think of an example where it wouldn't for a totally bounded set), certainly one can use the limit superior or the limit inferior to gain insight as to the "size" of a set.
- To calculate, really simply find a sequence of r-sizes going to 0, and calculate the r-capacities for this sequence. If the limit exists, then ANY sequence of r's going to 0, with their associated r-capacities will determine the same box dimension (Why?).

Example 3. Calculate $\mathrm{bdim}(I)$, for $I[0,1]$ with the metric d that I inherits from $\mathbb{R}$. Recall that if we were to use closed balls, then the $\frac{1}{2^n}$-capacity for I is $S_{(X,d)}\left(\frac{1}{2^n}\right) = 2^{n-1}$. But for open balls, we have $S_{(X,d)}\left(\frac{1}{2^n}\right) = 2^{n-1} + 1$. The box dimension should be the same for both. Indeed, it is: For the harder one,

$$\begin{aligned}
\mathrm{bdim}\,(I) &= \lim_{r \to 0} \frac{-\log S_{(X,d)}(r)}{\log r} = \lim_{n \to \infty} \frac{-\log\left(2^{n-1}+1\right)}{\log\left(\frac{1}{2^n}\right)} = \lim_{n \to \infty} \frac{\log\left(2^{n-1}+1\right)}{\log 2^n} \\
&\geq \lim_{n \to \infty} \frac{\log 2^{n-1}}{\log 2^n} = \lim_{n \to \infty} \frac{n-1}{n} = 1,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{bdim}\,(I) &= \lim_{r \to 0} \frac{-\log S_{(X,d)}(r)}{\log r} = \lim_{n \to \infty} \frac{-\log\left(2^{n-1}+1\right)}{\log\left(\frac{1}{2^n}\right)} = \lim_{n \to \infty} \frac{\log\left(2^{n-1}+1\right)}{\log 2^n} \\
&\leq \lim_{n \to \infty} \frac{\log 2^{n-1} \cdot n}{\log 2^n} = \lim_{n \to \infty} \frac{\log 2^{n-1}}{\log 2^n} + \lim_{n \to \infty} \frac{\log n}{\log 2^n} = \lim_{n \to \infty} \frac{n-1}{n} + \lim_{n \to \infty} \frac{\log n}{n} = 1 + 0 = 1.
\end{aligned}$$

Hence $\mathrm{bdim}\,(I) = 1$. Using the closed ball construction is even easier.

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Example 4. Let C be the Ternary Cantor Set. Show $\mathrm{bdim}(C) = \frac{\log 2}{\log 3}$. Here, assume that C sits inside I from the previous example, and again inherits its metric d from I. And since we can choose our sequence of r's going to zero, we will choose $r = \frac{1}{3^n}$, and consider only closed balls. Then one can show that $S_{(C,d)}\left(\frac{1}{2^n}\right) = 2^{n-1}$. (Think about this: At each stage, we remove the middle third of the remaining intervals. That means that at each stage we can cover each interval by a closed ball of radius $\frac{1}{3^n}$. This over covers the interval, but is not enough to cover two adjacent intervals. And since at each stage there are 2^{n-1} intervals, we are done. See the figure.

The calculation is now easy:

$$\begin{aligned}\mathrm{bdim}\,(C) &= \lim_{r\to 0} \frac{-\log S_{(C,d)}(r)}{\log r} = \lim_{n\to\infty} \frac{-\log\left(2^{n-1}\right)}{\log\left(\frac{1}{3^n}\right)} = \lim_{n\to\infty} \frac{\log\left(2^{n-1}\right)}{\log 3^n} \\ &= \lim_{n\to\infty} \frac{n-1}{n} \cdot \frac{\log 2}{\log 3} = \frac{\log 2}{\log 3}.\end{aligned}$$

In fact, we have the following:

Theorem 5. *Let $C \subset I$ be the Cantor set formed by removing the middle interval of relative length $1 - \frac{2}{\alpha}$ at each stage. Then*

$$\mathit{bdim}(C) = \frac{\log 2}{\log \alpha}.$$

A special note: All Cantor sets are homeomorphic. Yet, if we change the size of a removed interval at each stage, we effectively change the box dimension. This means that box dimension is NOT a topological invariant (remains the same under topological equivalence). Since a homeomorphism here would also act as a conjugacy between two dynamical systems on Cantor Sets, this also means that box dimension is also NOT a dynamical invariant.

2. Topological Entropy

For $f : X \to X$, a continuous map on a metric space (X, d), consider a sequence of new metrics on X indexed by $n \in \mathbb{N}$:

$$d_n^f(x, y) := \max_{0 \le i \le n-1} d\left(f^i(x), f^i(y)\right).$$

Here, the new metrics d_n^f actually measure a "distance" between orbit segments

$$\begin{aligned}\mathcal{O}_{x,n} &= \left\{x, f(x), \ldots, f^{n-1}(x)\right) \\ \mathcal{O}_{y,n} &= \left\{y, f(y), \ldots, f^{n-1}(y)\right)\end{aligned}$$

as the fartheset that these two sets diverge along the orbit segment, and assigns this distance to the pair x and y.

Exercise 1. Show for a given n that d_n^f actually defines a metric on X.

Now, using the metric d_n^f, we can define an r-ball as the set of all neighbor points y whose nth orbit-segment $\mathcal{O}_{y,n}$ stays within r distance of $\mathcal{O}_{x,n}$:

$$B_r(x, n, f) = \left\{ y \in X \,\middle|\, d_n^f(x, y) < r \right\}.$$

Convince yourself that as we increase n, the robit segment is getting longer, and more and more neighbors y will have orbit segments that move away from $\mathcal{O}_{x,n}$. Thus the r-ball will get smaller as n increases. But by continuity, the r-balls for any n will always be open sets in X that have x as an interior point. Also, as r goes to 0, the r-balls will also get smaller, right?

Now define the r-capacity of X, using the metric d_n^f and the new r-balls $B_r(x, n, f)$, denoted $S_{(X,d)}(r, n, f)$ (this is the SAME notion of r-capacity as the one we used for the box dimension! We are only changing the metric on X to d_n^f. But the actual calculations of the r-capacity depend on the choice of metric). As before, as r goes to 0, the r-balls shrink, and hence the r-capacity grows. And also, as n goes to ∞, we use the different d_n^f to measure ultimately the distances between entire positive orbits. This also forces the r-balls to shrink, and hence the r-capacity to grow. What is the exponential growth rate of the r-capacity as $r \to 0$? This is the notion of topological entropy:

Definition 6. Let $h_d(f, r) := \overline{\lim}_{n\to\infty} \dfrac{\log S_d(r, n, f)}{n}$. Then

$$h_d(f) := \lim_{r\to 0} h_d(f, r)$$

is called the *topological entropy* of the map f on X.

Next class, we will continue this discussion. But what you should take away from this part of the discussionis the following: Topological entropy is a measure of the tendency of orbits to diverge from each other. It will always be a non-negative number, and the higher it is, the faster orbits are diverging. Thus, this is a measure of the orbit complexity, and the higher the number, the more interesting (read messy) the dynamical structure.