HOMEWORK SET 9. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. General Information

The homework sets are listed here:

`http://www.mathematics.jhu.edu/brown/courses/s10/SyllabusS10421.htm`

2. Selected Exercises

Exercise (6.2.4). This problem can be solved by directly appeal to the equation of the ellipse. To keep the idea short, we will show only one aspect of this idea. Let an ellipse be centered at the origin with diameter on the horizontal axis, as pictured in Figure 2. Note that one can define the quantity $c = \sqrt{a^2 - b^2}$. The equation of this ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

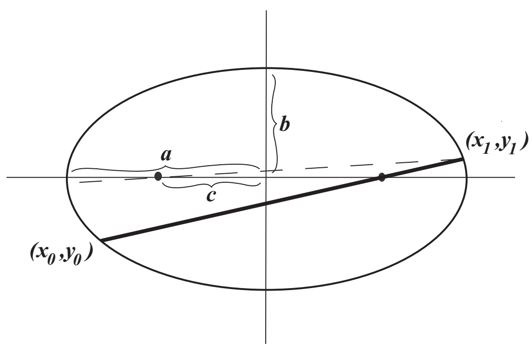

Figure 1. An orbit segment of an elliptic billiard table.

Now it should be understood that any orbit that starts on the ellipse and passes through a focus will rebound off the ellipse and immediate head for the other focus. This is part of the nature of the construction of an ellipse. Play the game and convince yourself that eventually, the orbit segments within the orbit inside the ellipse are straight lines all of which have the same sign slope. And also that the magnitude of these slopes converges to 0. In essence, then, a way to prove this result is to show that for each orbit segment, the magnitude of the y component of each line segment is smaller at the end of the segment than at the beginning, or that $|y_{i+1}| < |y_i|$. Then you would have to prove that the sequence of y-components converges to 0. We do the first part here:

Claim. *For the above line segment in the ellipse, $|y_1| < |y_0|$.*

Proof. First, understand that for $p_i = (x_i, y_i)$ to be on the ellipse, we can write it as $\left(-\frac{a}{b}\sqrt{b^2 - y_i^2}, y_i\right)$. (Just solve for x_i in terms of y_i and note the sign.) Now Calculate the equation of the line segment from p_0 through the right focus at $\left(\sqrt{a^2-b^2}, 0\right)$: The slope is

$$m_0 = \frac{-y_0}{\sqrt{a^2 - b^2} - \left(-\frac{a}{b}\sqrt{b^2 - y_i^2}\right)}.$$

Hence the equation of the line is

$$\begin{aligned} y - y_0 &= m_0 (x - x_0) \\ &= \frac{-y_0}{\sqrt{a^2 - b^2} + \left(\frac{a}{b}\sqrt{b^2 - y_0^2}\right)} \left(x + \frac{a}{b}\sqrt{b^2 - y_0^2}\right). \end{aligned}$$

Apply this to the point p_1 to get a comparison between y_0 and y_1:

$$y_1 - y_0 = \frac{-y_0}{\sqrt{a^2 - b^2} + \left(\frac{a}{b}\sqrt{b^2 - y_0^2}\right)} \left(\frac{a}{b}\sqrt{b^2 - y_1^2} + \frac{a}{b}\sqrt{b^2 - y_0^2}\right).$$

We now mess with this last equation:

$$\begin{aligned} y_1 - y_0 &= \frac{-y_0}{\sqrt{a^2 - b^2} + \left(\frac{a}{b}\sqrt{b^2 - y_0^2}\right)} \left(\frac{a}{b}\sqrt{b^2 - y_1^2} + \frac{a}{b}\sqrt{b^2 - y_0^2}\right) \\ &= \frac{-y_0 \frac{a}{b}}{\sqrt{a^2 - b^2} + \left(\frac{a}{b}\sqrt{b^2 - y_0^2}\right)} \left(\sqrt{b^2 - y_1^2} + \sqrt{b^2 - y_0^2}\right) \\ &< \frac{-y_0 \frac{a}{b}}{\frac{a}{b}\sqrt{b^2 - y_0^2}} \left(\sqrt{b^2 - y_1^2} + \sqrt{b^2 - y_0^2}\right) \\ &= \frac{-y_0}{\sqrt{b^2 - y_0^2}} \left(\sqrt{b^2 - y_1^2} + \sqrt{b^2 - y_0^2}\right) \\ &= -y_0 - y_0 \left(\frac{\sqrt{b^2 - y_1^2}}{\sqrt{b^2 - y_0^2}}\right). \end{aligned}$$

This leads to the inequality

$$\frac{y_1}{\sqrt{b^2 - y_1^2}} < \frac{y_0}{\sqrt{b^2 - y_0^2}}$$

which can be manipulated to produce

$$\begin{aligned} \frac{y_1^2}{b^2 - y_1^2} &< \frac{y_0^2}{b^2 - y_0^2}, \\ y_1^2 b^2 - y_1^2 y_0^2 &< y_0^2 b^2 - y_1^2 y_0^2, \\ y_1^2 b^2 &< y_0^2 b^2, \\ y_1^2 &< y_0^2, \end{aligned}$$

which implies $|y_1| < |y_0|$. $\square$

Exercise (6.4.1). The orbit that connects the four points where the symmetry axes cross the boundary is the period four orbit.

Exercise (EP28). Figure 6.2.2 on page 167 of the text is a basic drawing of the phase plane of the mathematical pendulum, albeit without the coordinate axes. I also drew it for you in class, along with the phase cylinder. The map between the phase plane and the cylinder is given by

$$(x, v) \mapsto \left(e^{2\pi i x}, v\right)$$

and makes sense only because the first coordinate is periodic. To derive the potential energy, simply take the original equation $\ddot{x} = -\sin(2\pi x)$ and realize that the force here is a gradient field. Thus $f = -\nabla V$ for some potential function $V(x)$. So $\sin(2\pi x) = \nabla V$ and

$$V = \int \sin(2\pi x) dx = -\frac{1}{2\pi}\cos(2\pi x),$$

and

$$H = \frac{1}{2}mv^2 - \frac{1}{2\pi}\cos(2\pi x).$$

The energy levels range from a global low of $H = -\frac{1}{2\pi}$ when both $x = v = 0$ on the cylinder (or $x = 0$, $v = 2n\pi$ for $n \in \mathbb{Z}$ in the plane) to no upper limit. However, on the level set $H = \frac{1}{2\pi}$ (on the cylinder), we have the saddle point $x = 0$, $v = \pi$, and two homoclinic orbits (they look heteroclinic in the phase plane). The orbit for points on these homoclinic orbits corresponds to an initial position and velocity perfectly tuned so that the pendulum swings toward the upper equilibrium solution asymptotically, never reaching it or ever stopping.

The closed orbits, both of lower energy than the saddle and of higher energy than the saddle (again on the cylinder), do NOT have the same period (contrast this with the linearization of the pendulum which is the harmonic oscillator. Now the question still remains as to why.

Exercise (EP29). Figure 2 shows that the radius of the caustic is $r = \cos\theta$. The equation for the circular caustic is then $x^2 + y^2 = r^2 = \cos^2\theta$.

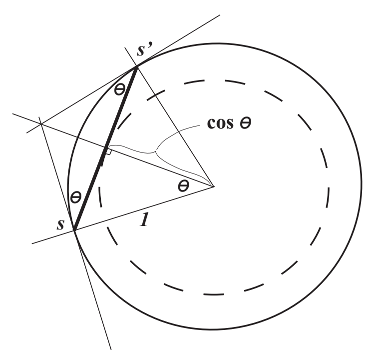

FIGURE 2. The caustic radius for the orbit of (s, θ) under the billiard map on the unit circle billiard table.

Exercise (EP30). By definition, a billiard on a circular table typically employs the "cyclic length parameter" s to define the phase space cylinder. For the unit circle, this has the effect that the cyclic length of the arc between two radial rays that differ by an angle 2θ is precisely 2θ. For a circle of radius r, the total length of the perimeter is $2\pi r$. Hence an arc cut out by two radial rays that differ in orientation by an angle 2θ is $2\theta r$ (See Figure 4). Thus the billiard map here is $\Phi(s,\theta) = (s + 2\theta r, \theta)$.

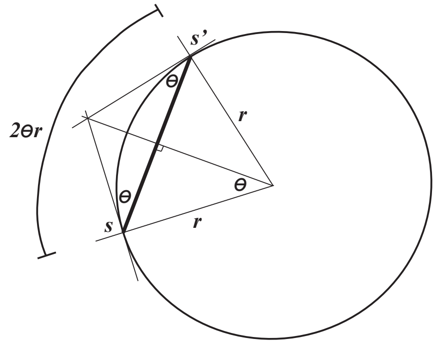

FIGURE 3. The arc length of the first return map of a circular billiard on a table of radius r.

Exercise (EP31). Figure 4 sets up the construction that makes this problem simple to understand. The two points to note are: 1) the quantity $r = \sin\theta$ and 2) the arc length from s to s' is $s' - s = 2\theta$. Put them together to get $r = \sin \frac{1}{2}(s' - s)$. Note that the Euclidean distance between s and s' is $2r$ and that the generating function is the negative of Euclidean distance and you are done. Finding the critical set is just Calculus I stuff. The set is any pair (s, s') where $s' = s + \pi$.

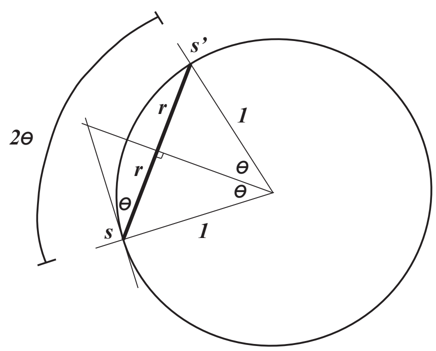

FIGURE 4. The chord length of the orbit segment from s to s' is $2r = 2\sin\theta$.