

HOMEWORK SET 8. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. GENERAL INFORMATION

The homework sets are listed here:

<http://www.mathematics.jhu.edu/brown/courses/s10/SyllabusS10421.htm>

2. SELECTED EXERCISES

Exercise (EP27). (a) There are many ways to do this. Simply build a metric on the torus, and show that the flow given by

$$\frac{dx}{dt} = \omega_1, \quad \frac{dy}{dt} = \omega_2$$

doesn't alter distance. Recall the flow is given by

$$T_\omega^t(x, y) = (x + \omega_1 t, y + \omega_2 t),$$

where $\omega = (\omega_1, \omega_2)$. One quick way to do this is to simply use the metric the torus inherits from the Euclidean metric on $\mathbb{R}^2$ via the map $\mathbb{R}^2 \mapsto \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$.

First, to understand this last statement, recall (page 63 in the text) the Euclidean metric on $\mathbb{R}$ induces a metric on the 1-torus, or S^1 . Indeed,

$$d_{S^1}([x], [y]) = \min \{d_{\mathbb{R}}(a, b) \mid a \in [x], b \in [y]\}.$$

This works also for the n -torus. Define an equivalence relation on $\mathbb{R}^n$ where $[(x_1, \dots, x_n)] = ([x_1], \dots, [x_n])$. This just means that two points in $\mathbb{R}^n$ correspond to the same point under the map

$$\exp : \mathbb{R}^n \rightarrow \mathbb{T}^n, \quad \exp(x_1, \dots, x_n) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$$

iff their vector difference is a vector of integers. See page 66 for some discussion. Then

$$d_{\mathbb{T}^n}([\vec{p}], [\vec{q}]) = \min \left\{ d_{\mathbb{R}^n}(\vec{a}, \vec{b}) \mid \vec{a} \in [\vec{p}], \vec{b} \in [\vec{q}] \right\}.$$

Now the calculation on $\mathbb{T}^2$ is easy: Let $\vec{p} = (x_p, y_p)$ and $\vec{q} = (x_q, y_q)$. Then

$$\begin{aligned} d_{\mathbb{T}^2}(T_\omega^t(\vec{p}), T_\omega^t(\vec{q})) &= \sqrt{((x_p + \omega_1 t) - (x_q + \omega_1 t))^2 + ((y_p + \omega_2 t) - (y_q + \omega_2 t))^2} \\ &= \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} \\ &= d_{\mathbb{T}^2}(\vec{p}, \vec{q}). \end{aligned}$$

Exercise (EP27). (b) Take any small rectangle on the cylinder whose sides are parallel to the two coordinate directions. The area of this rectangle is its length times its width, or its base times height. Upon applying the map, one retains a parallelogram, although the angles change. The area of a parallelogram is still the length of the base times the height, neither of which change under the map.

More to come....