HOMEWORK SET 7. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
PROFESSOR RICHARD BROWN

1. General Information

The homework sets are listed here:

`http://www.mathematics.jhu.edu/brown/courses/S10/SyllabusS10421.htm`

2. Selected Exercises

Exercise (4.3.3). Follow the solution in the back of the book. They are making the assumption that S^1 is parameterized by the unit interval, common to this text. However, is this still true if S^1 is parameterized by the interval $[0, \pi]$? $[0, 2\pi]$? Decide.

Exercise (4.3.9). A circle homeomorphism is necessarily one-to-one. Hence for a circle homeomorphism with finitely many fixed points, consider the sub-dynamical systems formed by restricting the map to the intervals between the fixed points. On each subinterval I_i bounded by a pair of adjacent fixed points, we will have either $f(x) > x$ on I_i or $f(x) < x$. On each I_i where $f(x) > x$, the left hand endpoint is a repeller on I_i and the right hand endpoint is an attractor. The opposite is true for the cases where $f(x) < x$. Since there exists an attractor (and finitely many fixed points in all), we will have both types of I_i. The attractor defines an actual crossing of the line $y = x$ (from above the line to below it), while for the adjacent intervals all on one side of the line $y = x$, the fixed points are semi-stable (draw some pictures). But since this is a circle map, we then must have a crossing from below the line $y = x$ to above it. Thus we must have a repeller also.

As for the last statement in the problem, consider the following map:

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{4} \\ 16\left(x - \frac{1}{2}\right)^3 + \frac{1}{2} & \frac{1}{4} \leq x \leq \frac{3}{4} \\ x & \frac{3}{4} \leq x \leq 1 \end{cases}$$

Here, there is an attractor (actually a super-attractor at $x = \frac{1}{2}$. But there are no repellers. There is also an infinite number of fixed points here. See Figure 1:

Exercise (EP24). The graph of this map is in Figure 2:

There are many interesting ways to study this map. For one: By the discussion at the top of page 127 in the text, if $x \in S^1$ is a period-q point of an orientation-preserving circle homeomorphism $f : S^1 \to S^1$, then any lift $F : \mathbb{R} \to \mathbb{R}$ satisfies $F^q(x) = x + p'$, $p' \in \mathbb{Z}$. Thus $\rho(F) = \frac{p'}{q}$ and modulus-1, $\rho(f) = \frac{p}{q}$, $p \in \{1, \ldots, q-1\}$. In fact, if q is the minimal period, then p does not divide q.

In our case, $x = 0$ is a period-4 point, with the orbit

$$0 \mapsto \frac{1}{3} \mapsto \frac{10}{27} \mapsto \frac{16}{27} \mapsto 1 = 0.$$

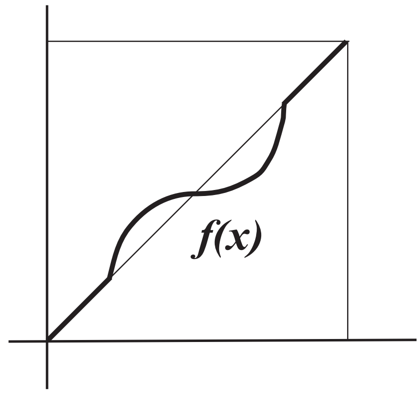

FIGURE 1. An S^1-map with an attractor and no repellers.

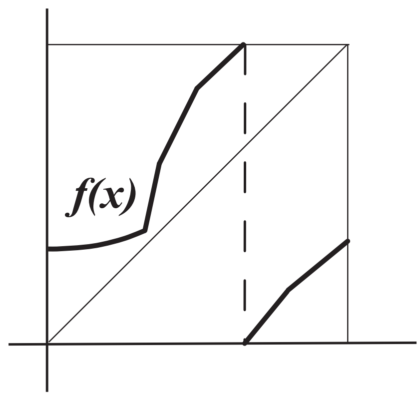

FIGURE 2. The complicated S^1-map $f(x)$ in **EP24**.

Thus, $\rho(f) = \frac{p}{4}$, where $p = 1$ or 3. But since the orbit above is monotonic, it must be the case that $p = 1$ (why?). Thus $\rho(f) = \frac{1}{4}$. Really, study the picture and think about it; The closest pure rotation for this map looks like a rotation by $\frac{1}{4}$ (think about straightening out the map to a pure rotation in a linear-regressive sort of way). This calculation simply verifies this.

Exercise (EP25). This is written out directly in the book and is the subject of **Remark 4.3.3**.

Exercise (EP26). A lift of any rotation map R_α is the map $F(x) = x + \alpha$. Really, a canonical one is when $\alpha \in [0, 1]$, but this is not really necessary.