HOMEWORK SET 5. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. General Information

The homework sets are listed here:

`http://www.mathematics.jhu.edu/brown/courses/S10/SyllabusS10421.htm`

2. Selected Exercises

Exercise (3.2.5). Here, you are looking for a linear map of $\mathbb{R}^2$ which comes from the time-1 map of an linear ODE system in the plane. The problem here is that the link between the two is the exponential map. Thus, the standard relationship between the eigenvalues of the ODE system at the origin and that of the linear map of $\mathbb{R}^2$ is that the latter is the exponential of the former. TO get negative eigenvalues for the time-1 map, we must be a bit more clever.

First, notice that a center of an ODE system corresponds to a rotation of the plane. In this situation, the ODE matrix A has purely imaginary eigenvalues. Call these $\pm\alpha i$. These are the roots of the characteristic equation corresponding to $A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$. The exponential matrix, then, will look like

$$e^{At} = \begin{pmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{pmatrix}.$$

Now, as a time-1 map, $e^A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, with characteristic equation $r^2 - 2r\cos\alpha + 1 = 0$. For this to have a single eigenvalue requires its discriminant to vanish: $4\cos^2\alpha - 4 = 0$. the only solution to this is for $\alpha = n\pi$, for $n \in \mathbb{Z}$. And since we also need the eigenvalue to be negative, we also need $\cos\alpha < 0$ (so that the characteristic equations will have negative roots). This leaves $\alpha = (2n+1)\pi$, $n \in \mathbb{Z}$. Choose $\alpha = \pi$. Then

$$e^{At} = \begin{pmatrix} \cos\pi t & \sin\pi t \\ -\sin\pi t & \cos\pi t \end{pmatrix},$$

so that the time-1 map is $e^A = \begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that the ODE equilibrium solution at the origin is a center, and motion is circular around it. Setting α to π means the time-1 map will take every vector to its negative, on the other side of the origin, exactly half way around the circle.

Exercise (EP17). **a**) The two eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ are $\lambda = \frac{1\pm\sqrt{5}}{2}$. This is a hyperbolic matrix, and the eigenvalue $\lambda = \frac{1-\sqrt{5}}{2}$ has modulus less than 1. Simply choose any starting vector in the eigenspace corresponding to this contracting eigenvector and its iterates under the powers of A will go to the origin. Indeed, the eigenvector equation is

$$A \cdot \vec{v} = \lambda \cdot \vec{v}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This boils down to choosing some vector along the line

$$v_2 = \frac{1-\sqrt{5}}{2} v_1.$$

Choose $v_1 = 1$ and $v_2 = \frac{1-\sqrt{5}}{2}$. if you take iterates under the powers of A, you will see that the orbit tends toward the origin exponentially (with what growth factor?).

One special thing to notice: The slope of the contracting eigenspace is irrational. Since it passes through the origin, there are no non-trivial rational starting vectors that would limit to the origin (you SHOULD be able to prove this readily by now). Thus the orbit of every non-trivial rational vector (vector with rational, non-zero elements) ultimately will diverge.

Exercise (EP17). c) Easily see is the first order vector recursion

$$\overrightarrow{u}_{n+1} = B\overrightarrow{u}_n, \text{ where } B = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}, \text{ and } \overrightarrow{u}_0 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

Exercise (EP17). d) The eigenvalues of $B = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$ are $\lambda = 1 + \sqrt{3}$ and $\mu = 1 - \sqrt{3}$. Again, this is a hyperbolic matrix. The formula from Proposition 3.1.13 is

$$a_n = \alpha v_1 \lambda^n + \beta w_1 \mu^n$$

where α, β, v_1, w_1 solve

$$\overrightarrow{u}_0 = \alpha \overrightarrow{v} + \beta \overrightarrow{w}, \tag{2.1}$$

and $\overrightarrow{v}$ and $\overrightarrow{w}$ are a choice of eigenvectors for λ and μ respectively. Here, we choose

$$\overrightarrow{v} = \begin{bmatrix} 1 \\ 1+\sqrt{3} \end{bmatrix}, \text{ and } \overrightarrow{w} = \begin{bmatrix} 1 \\ 1-\sqrt{3} \end{bmatrix}.$$

Then, Equation **??** is

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1+\sqrt{3} \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1-\sqrt{3} \end{bmatrix},$$

which we solve to find

$$\alpha = \frac{2+\sqrt{3}}{\sqrt{3}}, \quad \beta = \frac{\sqrt{3}-2}{\sqrt{3}}.$$

Thus

$$a_n = \frac{2+\sqrt{3}}{\sqrt{3}} \left(1+\sqrt{3}\right)^n + \frac{\sqrt{3}-2}{\sqrt{3}} \left(1-\sqrt{3}\right)^n.$$

One can now readily re-compute $a_0 = 2$, $a_1 = 6$, and $a_2 = 16$. To solve the last question, if 1980 is the zeroth year, with a population of only 2 lemmings, then 2010 is the 30th year, and the population this warm season would be $a_{30} = 26794772135936$. Really, though, in practice growth like this cannot be sustained at this level, no?

It is interesting also that for an equation so riddled with powers and quotients involving the irrational number $\sqrt{3}$, that the sequence takes integer values for all $n \in \mathbb{N}$.

Exercise (EP18). Let $\alpha \in \mathbb{Q}$. Then $\alpha = \frac{r}{n}$, where $r, n \in \mathbb{Z}$ are relatively prime. Then, for all $x \in S^1$, $R_\alpha^n(x) = x + n\alpha = x + r = x$. This shows that all points are n-periodic. Showing that there cannot be a point whose prime period is lower than n requires a bit more work. This is good enough for what I wanted to see.

In contrast, let $\alpha \notin \mathbb{Q}$ and assume there exists a periodic point x_0. Then there is a natural number n such that $R_\alpha^n(x_0) = x_0 + n\alpha = x_0$. But then $n\alpha = m \in \mathbb{Z}$, so that $\alpha = \frac{m}{n} \in \mathbb{Q}$. This contradiction establishes the fact that irrational rotations do not have periodic points.

Exercise (EP19). Here

$$\begin{aligned}\beta = 4\alpha - 1 \quad &= \quad \frac{4}{3 + \frac{1}{5 + \frac{1}{c}}} - 1 = \frac{4}{3 + \frac{1}{5 + \frac{1}{c}}} - \frac{3 + \frac{1}{5 + \frac{1}{c}}}{3 + \frac{1}{5 + \frac{1}{c}}} \\ &= \quad \frac{1 - \frac{1}{5 + \frac{1}{c}}}{3 + \frac{1}{5 + \frac{1}{c}}} = \frac{\frac{5 + \frac{1}{c} - 1}{5 + \frac{1}{c}}}{\frac{3\left(5 + \frac{1}{c}\right) + 1}{5 + \frac{1}{c}}} = \frac{4 + \frac{1}{c}}{16 + \frac{3}{c}} > \delta = \frac{1}{16 + \frac{3}{c}}.\end{aligned}$$

This last equality was calculated in the book on page 100.