HOMEWORK SET 4. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
PROFESSOR RICHARD BROWN

1. General Information

The homework sets are listed here:

`http://www.mathematics.jhu.edu/brown/s10/SyllabusS10421.htm`

2. Selected Exercises

Exercise (2.7.3). Let $x \in C$ where C is the ternary cantor set. Then there is a ternary expansion for x:

$$x = .\alpha_1\alpha_2\alpha_3\ldots = \sum_{i=1}^{\infty} \frac{\alpha_i}{3^i}, \text{ where } \alpha_i \neq 1. \tag{2.1}$$

Then, it is easy to see that $\frac{x}{3} \in C$ since

$$\frac{x}{3} = .0\alpha_1\alpha_2\alpha_3\ldots.$$

And then also

$$\frac{x}{3} + \frac{2}{3} = \frac{x+2}{3} = .2\alpha_1\alpha_2\alpha_3\ldots \in C.$$

Hence for $x \in C$, we have $f(x) \in C$, so that C is invariant under f. Measuring distance using the metric C inherits from $\mathbb{R}$, we have

$$d\left(f(x), f(y)\right) = \left|\frac{x+2}{3} - \frac{y+2}{3}\right| = \left|\frac{x-y}{3}\right| = \left|\frac{1}{3}\right| \cdot |x-y| = \frac{1}{3}d(x,y),$$

we see that f is indeed a $\frac{1}{3}$-contraction. What is the unique fixed point? There are no fixed points on the set $C \cap (0,1)$, and $f(0) = \frac{2}{3}$. And $f(1) = 1$. By the way, the number 1 is in C. Hence it has a ternary expansion, like in Equation 2.1. What is the ternary expansion for 1?

Exercise (3.1.5). Start drawing pictures of the first few values of n, and a clear pattern develops; At the nth stage, suppose you have k_n arrangements. Then for each nth stage arrangement, adding a 1×2 tile makes a new $(n+1)$th stage arrangement. However, all of the $(n-1)$th stage arrangements would also be $(n+1)$th stage arrangements with the addition of a 2×2 tile. Try some pictures and you will see that this would exhaust all of the $(n+1)$th stage arrangements. Thus we have the second-order recursion

$$k_{n+1} = k_n + k_{n-1}, \quad k_1 = 1, \quad k_2 = 2.$$

The solution is again the Fibonacci recursion, but with a different set of starting values.

This discussion rules out the idea of rotating any of the tiles. However, if we allowed rotations of tiles, then there are actually three arrangements for the 2×2

stage. To solve this problem play the same game: At the nth stage, suppose you have k_n arrangements. Then for each nth stage arrangement, adding a 1×2 tile makes a new $(n+1)$th stage arrangement. However, all of the $(n-1)$th stage arrangements would also be $(n+1)$th stage arrangements with the addition of a 2×2 tile. Also, in this case, though, each of the $(n-1)$th stage arrangements would be new $(n+1)$th stage arrangements with the addition of two 1×2 tiles oriented so that the short edges are adjacent to the $(n-1)$th stage arrangement. But any $(n+1)$th stage arrangement created from a $(n-1)$st stage arrangement with the addition of two 1×2 tiles oriented the other way will already be counted as coming from a nth stage arrangement. In this case, we wind up with the second-order recursion

$$k_{n+1} = k_n + 2k_{n-1}, \quad k_1 = 1, \quad k_2 = 3.$$

Incidentally, THIS is the answer discussed in the back of the book. Hence the book is wrong in stating that this is the solution not allowing tile rotations.

Exercise (EP13). To prove there are no continuous surjective contractions on S^1, let's assume it can happen and look for the contradiction. So suppose $f : S^1 \to S^1$ is a continuous, surjective contraction. Then, using either arc length or Euclidean length as our metric on S^1, there exists a $\lambda \in (0,1)$, where $\forall x, y \in S^1$,

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

By the Contraction Principle (Proposition 2.2.10), f has a unique fixed point, which we call x_0. Call $z \in S^1$ the antipodal point of x_0 (this is the point on the other side of the circle, or the other intersection of the standard unit circle in the plane with a line through the origin that passes through x_0). Since f is surjective, there exists $y \in S^1$ such that $f(y) = z$ (there may be more than one, but surjectivity tells us there has to be at least one such y). But then

$$d(z, x_0) = d(f(y), f(x_0)) \leq \lambda d(y, x_0)$$

absolutely cannot be satisfied since $d(z, x_0)$ is the maximum distance of any point from x_0. This fails to hold even if y is a fixed point so that $z = y$ (another contradiction to say there are two fixed points in a contraction). Hence the assumption that f is both a contraction and surjective is wrong.

To construct a non-surjective contraction is fairly straightforward:

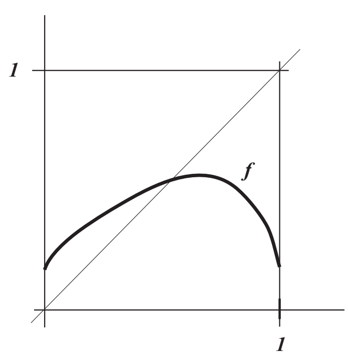

FIGURE 1. A contraction on S^1.

There are many ways to do this, and most involve the type of folding one sees in quadratic maps. For one, any function defined on the interval $[0,1]$ which takes values on $[0,1]$ can be viewed as a map (a continuous function) on S^1 as long as $f(0)=f(1)$. Create one with this condition (that is, whose graph lies completely within the unit square in $\mathbb{R}^2$, and has equal height at the endpoints), such that the graph cuts the fixed point line at one point in such a way that the fixed point ia an attractor, and you are done. A general example is given in the Figure 1. Notice here (construct some cobwebs) that the only fixed point in the figure is an attractor). For a more concrete expression for f, let

$$f(x)=\lambda x(1-x), \quad 0<\lambda<1,$$

be the logistic map. With this restriction on λ, the graph is on page 57 in the book.

Another method is to construct a map via an actual folding of the circle: Parameterize S^1 via the angular coordinate $\theta\in[-\pi,\pi]$, with the proper identification like we did in class. Then consider the map which folds the left-half of the circle (as the unit circle sits in the plane) onto the right-half (you should show analytically that this is a continuous map, although you can see it also). Indeed, using complex numbers:

$$f(e^{i\theta})=\begin{cases} e^{i\theta} & \text{if } \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ e^{i(\pi-\theta)} & \text{if } \theta\in\left[\frac{\pi}{2},\pi\right] \\ e^{-i(\pi+\theta)} & \text{if } \theta\in\left[-\pi,-\frac{\pi}{2}\right]. \end{cases}$$

Now simply compose this with the standard (half-angle) contraction on the right half-circle given by $g(e^{i\theta})=e^{i\frac{\theta}{2}}$. Then the composition function $(g\circ f): S^1\to S^1$ is a contraction, with attracting fixed point at $e^{i\cdot 0}=1$.

Exercise (EP14). The standard infinite cylinder is $C=S^1\times\mathbb{R}$. Since $2\cos 2x$ is π-periodic, simply scale the x-variable; let $s=\frac{x}{\pi}$. Then

$$g(s,y): C\to\mathbb{R}^2, \quad g(s,y)=\left(2\cos\frac{2}{\pi}s, 4y^2-y^4\right)$$

is a vector-valued function on C. To actually make it a map that takes values in C, one would have to rescale the output values to fit into the interval $[0,1]$. An obvious scaling would be

$$\widehat{g}(s,y): C\to C, \quad \widehat{g}(s,y)=\left(\frac{2+2\cos\frac{2}{\pi}s}{4}, 4y^2-y^4\right),$$

raising up the cosine function until all of it is above the horizontal axis and then dividing by the total range.

Now in the y-direction, $4y^2-y^4$ is certainly not periodic. But $\widehat{g}(s,0)=\widehat{g}(s,2)$. Hence

$$\widehat{g}(s,y)\big|_{y\in[0,2]}: [0,1]\times[0,2]\to[0,1]\times[0,4]$$

is a function on the rectangle given (but with values in a bigger rectangle!). Now again simply scale the y-variable; let $t=\frac{y}{2}$ and rescale the range in the second coordinate function to fit: Then

$$\bar{g}(s,y)\big|_{t\in[0,1]}: [0,1]\times[0,1]\to[0,1]\times[0,1], \quad \bar{g}(s,t)=\left(\frac{2+2\cos\frac{2}{\pi}s}{4}, \frac{16t^2-16t^4}{4}\right).$$

And since $\bar{g}$ is now a map on the unit square whose values on the two sides agree, one can identify the sides and $\bar{g}$ becomes a map on the torus.

Exercise (EP15a). Consider the continuous dynamical system given by the ODE system

$$\frac{d}{dt}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.2}$$

This is simply an uncoupled system $\dot{x} = ax$, $\dot{y} = by$ with solutions $x(t) = c_1 e^{at}$ and $y(t) = c_2 e^{bt}$ (you should be able to derive this), or

$$\vec{v}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{at} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{bt}.$$

The time-1 map is

$$x_0 = x(0) \mapsto x(1) = x_0 e^a$$
$$y_0 = y(0) \mapsto y(1) = y_0 e^b.$$

Hence the time-1 map for Equation 2.2 is the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\vec{v}) = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}\vec{v}$.

In our case, we have $\vec{x}_{n+1} = B\vec{x}_n$, a discrete dynamical system given by the matrix $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, where both eigenvalues are positive and less than 1. This system will match the time-1 map above when $\lambda = e^a$ and $\mu = e^b$ precisely, or $a = \log\lambda$ and $b = \log\mu$. Notice that this only works for $\lambda, \mu > 0$, but these are the only cases where the discrete system actually corresponds to the time-1 map of a continuous one! Hence, the ODE system whose time-1 map is given by B is

$$\frac{d}{dt}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \log\lambda & 0 \\ 0 & \log\mu \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Exercise (EP15b). Knowing the bouncing effects of a discrete linear dynamical system with negative eigenvalues (recall this from class?), you should be able to see that for $a, b \neq 0$, the above can be generalized to $a = \log|\lambda|$ and $b = \log|\mu|$ (the only thing that is different is the flipping we talked about in class). To find an expression for the solution curves, go back to the system given in Equation 2.2 above

$$\dot{\vec{v}} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\vec{v}$$

with solutions $x(t) = x_0 e^{at}$ and $y(t) = y_0 e^{bt}$. Solving for t in each and equating, we get

$$\frac{1}{a}\log\frac{x}{x_0} = \frac{1}{b}\log\frac{y}{y_0},$$
$$\log\left(\frac{x}{x_0}\right)^{\frac{1}{a}} = \log\left(\frac{y}{y_0}\right)^{\frac{1}{b}}.$$

Note here that we do not need to worry about the argument to the log function in this last expression. If we stay off of the axes, then all of $x, x_0, y, y_0 \neq 0$, and since x and x_0 (respectively y and y_0) are always of the same sign ($x = x_0 e^a$), the argument to log is always positive.

This last equation immediately implies (via the exponential map) that

$$\left(\frac{x}{x_0}\right)^{\frac{1}{a}} = \left(\frac{y}{y_0}\right)^{\frac{1}{b}}.$$

Allowing for λ and/or μ to be negative only will involve flipping of the discrete orbit across an eigenline. To include this type of behavior, we throw in the absolute values to include solution lines which are reflections across both eigendirections so that

$$\left|\frac{x}{x_0}\right|^{\frac{1}{a}} = \left|\frac{y}{y_0}\right|^{\frac{1}{b}}.$$

Now simply solve for $|y|$ to get

$$|y| = |y_0| \left|\frac{x}{x_0}\right|^{\frac{b}{a}} = C\,|x|^{\alpha},$$

where $C = \frac{|y_0|}{|x_0|^{\alpha}}$, and $\alpha = \frac{b}{a} = \frac{\log|\lambda|}{\log|\mu|}$.