

HOMWORK SET 3. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. GENERAL INFORMATION

The homework sets are listed here:

<http://www.mathematics.jhu.edu/brown/s10/SyllabusS10421.htm>

2. SELECTED EXERCISES

Exercise (2.4.6). This is really just basic ODE theory: If an autonomous, first order ODE $\dot{x} = f(x)$ has an isolated equilibrium solution at x_0 , and $f'(x_0) < 0$, then the equilibrium solution is a sink. The reasoning goes that for any $x < x_0$ but sufficiently close, the unique solution $x(t)$ passing through x at time-0 will always have $\dot{x}(t) > 0$ for all $t > 0$, since here $f(x) > 0$ for $x < x_0$ and close by. On the other side of x_0 , at least nearby, $\dot{x} = f(x) < 0$. Hence solutions that start on this side of x_0 will decay to x_0 .

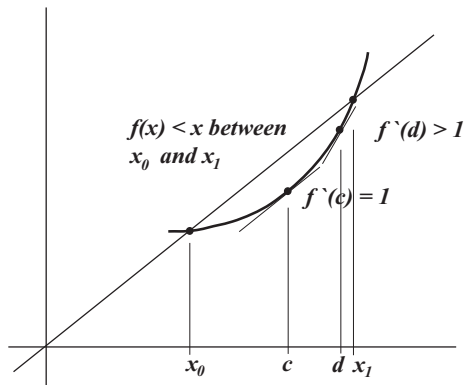
In a more general sense, a system of first order ODEs $\dot{\vec{x}} = f(\vec{x})$ which is “nice” near a fixed point $\vec{x}_0$ has a special quality near that fixed point. This special quality is that, under certain non-degeneracy conditions, the behavior of solutions on the original nonlinear system is qualitatively the same as that of the linear system formed by the first Taylor approximation to the right hand side f at the point $\vec{x}_0$ (this is the essence of the term *local linearization*). The term “nice” can be defined as “ $f(x)$ is continuously differentiable in a neighborhood of x_0 ”. For a 1×1 system, the original equation can be linearized around x_0 , since $f'(x_0)$ exists, and as long as $f'(x_0) \neq 0$, the solution near x_0 of the original equation will behave much like that of the linear equation.

Given that discussion, the linearized equation is

$$\dot{x} = f'(x_0)x = kx, \quad \text{where } k < 0.$$

Solutions to this equation are $x(t) = Ce^{kt}$, and with $k < 0$, and all nearby solutions exponentially decay to 0 exponentially. Thus the original system has the same property asymptotically at x_0 . Indeed, this linear system has a linear time-1 map $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = e^k x$. This is obviously a e^k -contraction since $k < 0$. Since the vector field $f(x)$ is continuously differentiable, the original ODE will be a e^k -contraction asymptotically at x_0 . Hence, there will assuredly be a small interval around x_0 where the ODE will remain a contraction. Hence, on this interval, all solutions will converge exponentially to x_0 .

Exercise (2.5.3). Suppose the fixed point set $Fix(f)$ is not connected. Then there must be a gap between two points in the fixed point set where no other fixed points reside. Call this gap the open interval (x_0, x_1) where $x_0, x_1 \in Fix(f)$ $x_1 - x_0 > 0$ (assuming $x_1 > x_0$) and $\forall x \in (x_0, x_1)$, $x \notin Fix(f)$.



Then it must be the case that for all $x \in (x_0, x_1)$, either $f(x) > x$ or $f(x) < x$ (the graph of f either lies completely above the diagonal in the unit square, or completely below it). Suppose for instance, $f(x) < x$, for all $x \in (x_0, x_1)$. By the Mean Value Theorem, there must exist a point $c \in (x_0, x_1)$ where

$$f'(c) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = 1.$$

However, now consider the interval (c, x_1) . Again by the Mean Value Theorem, there must exist a point $d \in (c, x_1)$, where

$$f'(d) = \frac{f(x_1) - f(c)}{x_1 - c} > \frac{x_1 - c}{x_1 - c} = 1$$

since here $f(c) < c$. But this is impossible, because by supposition, on all of $[0, 1]$, we have $|f'(x)| \leq 1$. Thus, by contradiction, there can be no gap between the fixed points x_0 and x_1 where the graph of f lies completely below the diagonal.

The proof in the other case, where $f(x) > x$ for all $x \in (x_0, x_1)$ is entirely symmetric and is left to you.

Exercise (2.5.4). Basically, see the solution to **2.3.2**. With the conditions given, the square of the map (the *composite* square) will have a derivative such that $0 \leq (f \circ f)'(x) < 1$. Thus the square of f will be a non-decreasing function which will have only fixed points and points asymptotic to fixed points. Thus f can have only fixed and possible period-2 points.

Exercise (EP8). The general solution to this uncoupled system is

$$\begin{aligned} r(t) &= \frac{r_0}{r_0 + (1 - r_0)e^{-\frac{t}{2}}} \\ \theta(t) &= t + \theta_0 \\ z(t) &= z_0 e^{-t}. \end{aligned}$$

Along the circle given by the two equations $r = 1$ and $z = 0$ (remember from Calculus III that the typical 1-dimensional set in $\mathbb{R}^3$ requires two equations), the motion is constant velocity. More precisely, the solution passing through the initial point $\vec{x} = (r_0, \theta_0, z_0) = (1, 0, 0)$ is given by

$$r(t) = 1, \quad \theta(t) = t, \quad z(t) = 0.$$

Since by observation, there are no other cycles near this one (or anywhere else, for that matter), this is an isolated cycle, with period 2π .

Choose the plane passing through the point $(1, 0, 0)$, and normal to the motion along the cycle; the tangent vector to the motion passing through $(1, 0, 0)$ is the tangent vector $(0, 1, 0)$. Note that with cylindrical coordinates, we only view this plane as a half-plane, to distinguish it from the plane as that described by the equation $\theta = \pi$. Call this plane $\mathcal{P}_\theta$.

Every orbit that starts in this plane (even the origin) intersects this plane again at $t = 2\pi$. Using (r, z) as coordinates on $\mathcal{P}_\theta$, we get a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (actually it is a function on the half-plane defined by $r \geq 0$ and z , but near the point $(r, z) = (1, 0)$, this will not matter) defined by the condition that $(r_0, z_0) = (r(0), z(0)) \mapsto (r(2\pi), z(2\pi))$. Thus

$$f(r, z) = \left(\frac{r}{r + (1 - r)e^{-\pi}}, \quad ze^{-2\pi} \right).$$

Notice that the point $(1, 0)$ is fixed by f and that f is nonlinear (in r only, though). However, to understand the qualitative behavior of f near the fixed point at $(1, 0)$, we calculate the derivative map there:

$$Df_{(1,0)} : \left(\begin{array}{cc} \frac{\partial f_r}{\partial r} & \frac{\partial f_r}{\partial z} \\ \frac{\partial f_z}{\partial r} & \frac{\partial f_z}{\partial z} \end{array} \right) \bigg|_{(1,0)} = \left(\begin{array}{cc} e^{-\pi} & 0 \\ 0 & e^{-2\pi} \end{array} \right).$$

It is immediate that all of the eigenvalues of the fixed point $(1, 0)$ under the map f are positive and strictly less than 1. Hence this orbit is an attractive orbit and a limit cycle.

Exercise (EP9). Really, for λ outside of the interval $[0, 4]$, the range of f as a real-valued function includes points outside of $[0, 1]$.