HOMEWORK SET 2. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. General Information

The homework sets are listed here:

`http://www.mathematics.jhu.edu/brown/courses/s10/SyllabusS10421.htm`

2. Selected Exercises

Exercise (2.2.5). There are a number of ways to find the second-order recursion that defines the population of lemmings. For one, start from the bottom and work your way up. Consider it this way: Take you census every spring. Then last year's summer litter is maturing and the 2 year-olds have just recently died. In year 1, we have just the one pair, and they are preparing to have young. They will have 4 young and the next winter, none will die. The next spring (year 2), there are 6 now, but the original pair is turning 2 this summer. They all produce, and the 12 new young make the fall population 18. The original 2 die in the winter, and the next spring (year 3), there are 16. More precisely, let y_n = spring population in year n, b_n the summer population 3 months later, and d_n the winter deaths 6 months after that. Then it is easy to see that

$$y_{n+1} = y_n + b_n - d_n.$$

However, there are relationships between these variables. For one, every spring pair produces four young in the summer. Thus

$$b_n = \frac{1}{2} y_n \cdot 4 = 2y_n.$$

Next, the number of deaths in any winter is precisely the number of births 2 summers before the previous spring (lemmings die in the winter of their third year). Thus

$$d_n = b_{n-2}.$$

Thus, up to now, we get

$$\begin{aligned} y_{n+1} &= y_n + 2y_n - b_{n-2} \\ &= 2y_n + (y_n - b_{n-2}). \end{aligned}$$

The last line is written this way because it is also true that the population in any spring minus the births two years before is precisely the number of births in the intermediate year, which is precisely twice the spring population of that year. Indeed,

$$y_n - b_{n-2} = b_{n-1} = 2y_{n-1}.$$

Putting this in for the term in the parentheses, we get

$$y_{n+1} = 2y_n + 2y_{n-1}$$

and we are done.

Notice that the book is incorrect in its solution to this problem. However, the rest of the problem is fine. Let $x_{n+1} = \frac{y_{n+1}}{y_n}$. Recall that for a second order recursive sequence is asymptotically exponential, with a growth rate given by the limit of the sequence $\{x_n\}$. TO find this, notice that

$$x_{n+1} = \frac{y_{n+1}}{y_n} = \frac{2y_n - 2y_{n-1}}{y_n} = 2 + 2\frac{y_{n-1}}{y_n} = 2 + \frac{2}{x_n}.$$

If we let $g(x) = 2 + \frac{2}{x}$, then $x_{n+1} = g(x_n)$. This is a contraction on a suitable interval like $I = [2, 4]$, and all orbits are asymptotic to the fixed point at $x = 2 + \frac{2}{x}$, which is $x^2 - 2x - 2 = 0$ (notice this similar structure of this last equation and the recursive y-sequence that defined it). The only root to this in I is $x = 1 + \sqrt{3}$.

One final note here; there are other ways to do this. One is to count lemmings each year by age. To do this every spring, simply note that every spring only the 0-year olds and the 1-year olds are to be counted, because the two year olds in that spring are actually $2\frac{3}{4}$ years of age, and have died before getting there. To get the living lemming counts, just follow the number of births each year. You still have to work out the recursion formula, but the sequence is simple to calculate.

Exercise (2.2.6). The conditions here stipulate that x is a fixed point with a unit slope in absolute value and that x is not an inflection point. One case of this is that the graph of $f(x)$ is tangent to the fixed point line (the line $y = x$ which we call ℓ) and the concavity of f does not change here (that is, $f''(x) \neq 0$). One conclusion in this case is that near x (except at x), either the graph of f is completely above ℓ or completely below it (draw pictures). Indeed, consider the former. Let $U = B_\epsilon(x)$ be a small neighborhood around x, where $\forall y \in U$, $y \neq x$, we know $f(y) > y$. But then for $y > x$, we have $f(y) > f(x) = x$. This says the elements of the orbit of y, $\mathcal{O}_y$ are not moving towards x. Once one realizes that to converge to x, the elements of $\mathcal{O}_y$ would have to jump to the other side of x. But they cannot because then $f(x)$ would have to dip back below the fixed point line ℓ, creating another fixed point which $\mathcal{O}_y$ would then converge to. The other case is similar.

I leave the case where $f'(x) = -1$ to you.

Exercise (2.3.2). Since $f : [0, 1] \to [0, 1]$ is non-increasing, for $x_0, y_0 \in [0, 1]$, if $y_0 > x_0$, then $f(y_0) \leq f(x_0)$. Rewritten, we have for $y_0 > x_0$, $f(y_0) = y_1 \leq x_1 = f(x_0)$. And since $x_1 \geq y_1$, we get $f(x_1) = x_2 \leq y_2 = f(y_1)$. Put these together to get

$$\text{if } y_0 > x_0, \text{ then } f\left(f(y_0)\right) \geq f\left(f(x_0)\right).$$

That is, the map $f^2 : [0, 1] \to [0, 1]$ is a non-decreasing map. By Proposition 2.3.5, all $x \in [0, 1]$ are either fixed points or asymptotic to fixed points. This automatically rules out any higher order periodic points for f^2 (Why?). Thus, f^2 can have only fixed points, which means that f can periodic points of order at most 2.

Example. Let $f(x) = 1 - x$. Here all points are of order-2 except for $x = \frac{1}{2}$. Also for $g(x) = 1 - x^2$, the end points are of period 2 and the only fixed point is $\frac{\sqrt{5}-1}{2}$. What can you say about the dynamics of $g(x)$?

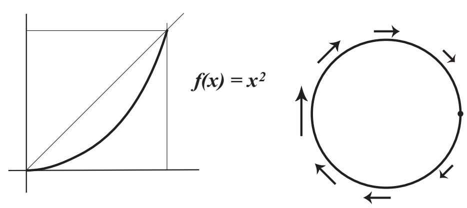
Exercise (EP5). The model for this function is given in the book: Example 2.2.4 and the function $\sqrt{x}$ on $[1, \infty)$. In our case, with $f(x) = \log(x-1)+5$ on $I = [2, \infty)$, we need a slight modification. Here, f is increasing on all of I, and differentiable, with $f'(x) = \frac{1}{x-1}$. However, by Proposition 2.2.3, we only know here then that $f(x)$ is 1-Lipschitz, since $f'(2) = 1$. Thus, we cannot say directly that f is a contraction. To cure this, notice that $f(2) = 5$ and indeed $f(I) = [5, \infty)$. Hence, after one iteration of f, we can regard f as a map on $[5, \infty)$. Here, since $f'(5) = \frac{1}{4}$, it is easy to see that f is a $\frac{1}{4}$-contraction (on $[5, \infty)$ only, but remains a contraction throughout I in this case), and hence has a global attractor. This immediately rules out any periodic points of order higher than 1. To find the only fixed point, simply solve $x = f(x)$, or take any starting point and iterate. It is approximately 6.74903 using the natural log and 5.66925 using the common log.

Exercise (EP6). Let $I = [a, b]$. For part a), make the assumption that the invertible map $f : I \to I$ is not one-to-one (injective). Then $\exists x, y \in I$ such that $x \neq y$ but $f(x) = f(y)$. Since $f(x)$ is invertible, f^{-1} is a function. But then

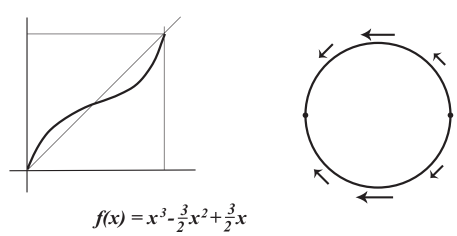
$$x = f^{-1}(f(x)) = f^{-1}(f(y)) = y$$

is a contradiction. Hence the assumption must be wrong. The other two parts are constructed similarly.

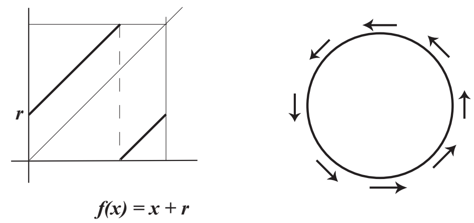
Exercise (EP7). To answer the first request, take any increasing function $f : I = [0, 1] \to I$ that fixes both endpoints, and you are done. This will correspond to an injective continuous function on S^1. Note the fact that it is increasing allows that the inverse exists, and the fact that the endpoints are fixed makes the inverse continuous. For an example, the function $f(x) = x^2$ is a continuous invertible function on I which induces a continuous, invertible function on S^1 (What is an expression for the inverse?) See the figure:



Really, the end point need not be fixed, as long as $f(0) = f(1)$. For a differentiable circle map, the one-sided derivatives of the map f at the endpoints must agree in addition to the map being continuous. Try $f(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}$, as in the next figure:



Here, considered as a function on $\mathbb{R}$, $f'(0) = f'(1)$. Finally, since the graph of $f(x)$ on I need not actually be in I due to the fact that we are identifying the endpoints of I, it is quite easy to create a differentiable function on the circle that comes from a function whose domain is I but whose range is shifted a bit. This leads to an easy to describe differentiable fixed point free map on the circle $f(x) = x + r$, where $r \in I$, as in the figure:



Note that this only works due to the fact the via the identification of the endpoints, one can "wrap" the graph vertically. You will see these kind of graphs again in circle , cylinder and toral maps later.