HOMEWORK SET 1. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. General Information

The homework sets are listed here:

`http://www.mathematics.jhu.edu/brown/SyllabusS09421.htm`

Claim. *For $x \in I = [1, \infty)$ and $t \geq 0$, $\sqrt{x+t} \leq \sqrt{x} + \frac{1}{2}t$.*

Note: There is a typo in the book when they establish this fact (can you find it?). Here we this with a bit of Calculus II. Really, this only involves a statement that the tangent line to the graph of $\sqrt{x}$ lies above the graph for $x \in I$.

Proof. Let $g(t) = \sqrt{x+t}$ for some fixed $x \in I$ defined on $t \geq 0$ (really, we need $t \in (-\epsilon, \infty)$ for small $\epsilon > 0$ since we will need the differentiability g at $t = 0$).

Expand $g(t)$ about $t = 0$ as a Taylor Series:

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2}t^2 + \cdots.$$

Here all of the derivatives of $g(t)$ at 0 are defined and positive, so for $t \geq 0$, $g(t) \leq g(0) + g'(0)t$ (this is what I mean by the tangent line lying over the graph). Hence

$$\sqrt{x+t} = g(t) \leq g(0) + g'(0)t = \sqrt{x} + \frac{1}{2\sqrt{x}}t \leq \sqrt{x} + \frac{1}{2}t$$

since $\sqrt{x} \geq 1$ on I. □

2. Selected Exercises

Exercise (2.1.1). Since $x_{n+1} = kx_n + b$, under the coordinate transformation $y = x - \frac{b}{1-k}$, we get

$$\begin{aligned}
y_{n+1} &= x_{n+1} - \frac{b}{1-k} \\
&= (kx_n + b) - \frac{b}{1-k} \\
&= k\left(y_n + \frac{b}{1-k}\right) + b - \frac{b}{1-k} \\
&= ky_n + \frac{kb}{1-k} + b - \frac{b}{1-k} \\
&= ky_n + \frac{kb + b(1-k) - b}{1-k} = ky_n.
\end{aligned}$$

Exercise (EP2). The general solution to the ODE should be readily computable, and is

$$p(t) = 900 + Ce^{\frac{t}{2}}.$$

Constructing a discrete dynamical system for the time-1 map is a little more work. You are looking for a map $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\begin{aligned} p(0) &\overset{f}{\mapsto} p(1) \\ 900 + C &\overset{f}{\mapsto} 900 + Ce^{\frac{1}{2}}. \end{aligned}$$

The variable C, while a good parameter for the solution curves of the ODE, does not match with the standard coordinate system on $\mathbb{R}$. Hence change the variable: Set $t = 900 + C$. Then $C = t - 900$, and

$$f(t) = 900 + (t - 900)e^{\frac{1}{2}} = e^{\frac{1}{2}}t + 900\left(1 - e^{\frac{1}{2}}\right).$$

This is a linear function in t and the dynamical system on $\mathbb{R}$ with this f has a unique fixed point at $t = 900$. So now, what are the dynamics of this system?

As a further exercise, this map is invertible (why?). Show that the inverted system (the dynamical system based on f^{-1} is a contraction (what is the Lipschitz constant for f^{-1})

Exercise (EP3). Let $f(x) = \sqrt{x}$ for $x \in I$ defined in the claim above. Choose two points x, $y \in I$ and suppose without loss of generality that $y - x = t \geq 0$. Then $y = x + t$ and

$$d(y, x) = |x + t - x| = t.$$

For the images, we have $d(f(y), f(x)) = \sqrt{x+t} - \sqrt{x}$. By the claim, we know that $\sqrt{x+t} - \sqrt{x} \leq \frac{1}{2}t$ for $x \in I$. Thus

$$d(f(y), f(x)) = \sqrt{x+t} - \sqrt{x} \leq \frac{1}{2}t = \frac{1}{2}d(y, x).$$

Hence, $f(x)$ is a $\frac{1}{2}$-contraction on I.

Exercise (EP4). Since $f : X \subset \mathbb{R}^n \to X$ is a λ-contraction (i.e., f is λ-Lipschitz with $\lambda < 1$),

$$\forall x, y \in X, \quad d(f(x), f(y)) \leq \lambda d(x, y).$$

Suppose $x \in X$ and $x \neq y \in X$ are two distinct fixed points, so that $f(x) = x$, and $f(y) = y$. Then

$$d(f(x), f(y)) = d(x, y) \not\leq \lambda d(x, y)$$

for any $\lambda < 1$.