HOMEWORK SET 10. SELECTED SOLUTIONS

DYNAMICAL SYSTEMS (110.421)
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1. General Information

The homework sets are listed here:

http://www.mathematics.jhu.edu/brown/courses/s10/SyllabusS10421.htm

2. Selected Exercises

Exercise (7.1.1). Fix $m \geq 2$ (the minus sign won't matter but ignoring it will make the proof easier), so that $E_m : x \mapsto mx \mod 1$. Let $x = \frac{p}{q}$ be rational. Then

$$\mathcal{O}_x^+ = \left\{ \frac{p}{q}, \frac{mp}{q} \mod 1, \ldots, \frac{m^n p}{q} \ldots \right\}.$$

Here the numerator $m^n p \in \mathbb{Z}$ and modulo q, it must be the case that $\forall n \in \mathbb{N}$

$$m^n p \mod q \in R = \{0, 1, \ldots, q-1\}.$$

suppose x is not eventually periodic. Then for each n, $m^n p$ corresponds to a unique element in the above set. But for $n > q$, it must be the case that either there are two distinct values $n_1 \neq n_2$ where $m^{n_1} p = m^{n_2} p$ modulo q are the same (correspond to the same element in R), or else $m^n p = 0$ modulo q. In either case, the forward orbit from this point on is periodic.

Exercise (7.1.4). Suppose not. Then $\exists n \in \mathbb{N}$ such that $P_n(f)$ consists of periodic points all of whose periods are of minimal period some m where $m \,|\, n$. By Proposition 7.1.4, $P_i(f) = 2^i$, $\forall i \in \mathbb{N}$. But

$$\sum_{i=1}^{n-1} P_i(f) = \sum_{i=1}^{n-1} 2^i = 2^n - 1 < P_n(f).$$

As $P_n(f) = 2^n$ this cannot be the case, and the assumption cannot be true.

Exercise (7.1.6). Choose any logistic map f_λ where $\lambda \in (0, 1]$.

Exercise (EP32). The period-2 points of the map $f : S^1 \to S^1$ given by $f(z) = z^2$ (thinking of the circle as the unit modulus complex numbers), satisfy $(z^2)^2 = z^4 = z$, or $z^3 = 1$. Here we are looking for the cubic roots of unity, namely

$$z = e^{2\pi i \left(\frac{k}{3}\right)}, \quad k = 1, 2, 3.$$

In general, the periodic points will satisfy the equation

$$z^{2^n} = z, \qquad \text{or } z^{2^n - 1} = 1,$$

so that the n-periodic points are the $(2^n - 1)$th roots of unity. So the 3-period points are

$$z = e^{2\pi i \left(\frac{k}{7}\right)}, \quad k = 1, \ldots, 7,$$

and the period-4 points are

$$z = e^{2\pi i\left(\frac{k}{15}\right)}, \quad k = 1, \ldots, 15.$$