HOMEWORK PROBLEM SET 4: DUE SEPTEMBER 27, 2019

AS.110.202 CALCULUS III
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The following problem set is based on **Sections 2.3**, **2.4**, and **2.5** of the text. Along with the exercises below, please do the following:

- **WeBWorK:** Complete Problem Set 4 on WeBWorK.
- **Reading for next week:** Reread **Section 2.5**, and read **2.6**, **3.1** and **3.2**.

For practice (neither to be handed in nor graded), here is a set of selected textbook problems:

- **Section 2.3:** 3acde,5,9,13,14,19,21,26
- **Section 2.4:** 2,4,6,12,13,15,18,21,24,25
- **Section 2.5:** 3a,5,6,8,9,10,11,18,32,35,36

The three following exercises are to be handed in for grade in lecture on the due date above:

Exercise 1. Do the following:

(a) Write a statement defining the Chain Rule for the functions $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{f}\colon \mathbb{R}^m \to \mathbb{R}^p$. Then describe how it works in a paragraph, assuming the reader is a classmate who has been following the course but missed the lecture on Properties of the Derivative.

(b) Explain in detail how the Chain Rule you learned in Calculus I,

$$(f \circ g)'(x) = f\left(g(x)\right) \cdot g'(x),$$

is really just the special case of your statement in **part (a)** when $m = n = p = 1$.

(c) Consider a path $\mathbf{c}\colon [a,b] \to \mathbb{R}^n$ and a differentiable function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$. Assume that for every $a < t < b$, $\mathbf{c}'(t) \neq 0$. Use the Chain Rule to find a vector that is tangent to the curve parameterized by the composite path $(\mathbf{f} \circ \mathbf{c})\colon [a,b] \to \mathbb{R}^m$ at the point $\mathbf{f}\left(\mathbf{c}(t_0)\right)$, for $t_0 \in (a,b)$.

(definitions, derivatives, viewpoint)

Exercise 2. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a real-valued function on $\mathbb{R}^n$, and $\mathbf{x}_0 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ a point.

(a) If f is differentiable at $\mathbf{x}_0$, explain what it means to say that the function $z = h(\mathbf{x})$, where

$$h(\mathbf{x}) = f(\mathbf{x}_0) + Df(\mathbf{x}_0)\,(\mathbf{x} - \mathbf{x}_0)$$

is the best linear function to approximate f at $\mathbf{x}_0$.

(b) Suppose f is differentiable at the point $\mathbf{x} = \mathbf{x}_1$, and $Df(\mathbf{x}_1) = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$, the $\mathbf{0}$-matrix. Describe what may be happening to $\mathbf{graph}(f)$ at or near $\mathbf{x} = \mathbf{x}_1$.

(c) Suppose f is not differentiable at $\mathbf{x} = \mathbf{x}_2$. At this point, all of the partial derivatives exist and are continuous as functions, except for one: In the ith coordinate direction, we found

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_2) = \lim_{h \to 0} \frac{f(\mathbf{x}_2 + \mathbf{e}_i h) - f(\mathbf{x}_2)}{h} = \infty.$$

Describe what may be happening to $\mathbf{graph}(f)$ at or near $\mathbf{x} = \mathbf{x}_2$.

(differentiability, linear approximation, shape of graphs)

Exercise 3. The Product Rule, on page 125 of the text, is only defined for real-valued functions $f, g \colon U \subset \mathbb{R}^n \to \mathbb{R}$ since the product is the scalar product $f(\mathbf{x})g(\mathbf{x})$. However, when it makes sense, it also works for vector-valued functions. As an example, let $\mathbf{c} \colon [a,b] \to \mathbb{R}^n$ and $\mathbf{d} \colon [a,b] \to \mathbb{R}^n$ be two differentiable paths in $\mathbb{R}^n$, and define the real-valued function $h \colon [a,b] \to \mathbb{R}$ by $h(t) = \mathbf{c}(t) \bullet \mathbf{d}(t)$ using the Dot Product in $\mathbb{R}^n$. First, explain why if both $\mathbf{c}$ and $\mathbf{d}$ are differentiable, then so is h. Then, calculate $h'(t)$ in terms of $D\mathbf{c}(t)$ and $D\mathbf{d}(t)$, and show that it satisfies the Product Rule

$$\frac{dh}{dt}(t) = D\mathbf{c}(t) \bullet \mathbf{d}(t) + \mathbf{c}(t) \bullet D\mathbf{d}(t).$$

(derivatives, product rule, differentiability)