HOMEWORK PROBLEM SET 3: DUE SEPTEMBER 20, 2019

AS.110.202 CALCULUS III
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Question 1. Do WeBWorK Problem Set 3.

Question 2. For $f : X \subset \mathbb{R}^n \to \mathbb{R}$, with coordinates $(x_1, x_2, \ldots, x_i, \ldots, x_n)$ in $\mathbb{R}^n$, do the following:

(a) Define and describe the graph of f in terms of its properties like in which space is it located, how one envisions the domain and codomain of f in the space where it is located, and what information it conveys.

(b) For a fixed real number $c \in \mathbb{R}$, use the graph of f to describe the c-level set of f. (Do keep in mind that the graph of f and the level sets of f exist in Euclidean spaces of different dimensions.) In your description, discuss the possible sizes, or dimensions of a level set and where the level sets reside. Hint: First think of the case where $n = 2$ and then generalize.

(c) For a fixed real number $a \in \mathbb{R}$, use the graph of f to describe the $(x_i = a)$-vertical slice of the graph of f (the book, on page 80, calls these slices *sections*.) In your description, discuss the possible sizes or dimensions of this slice, where is resides, and how it is situated with regard to the graph of f.

(d) Now create a special slice through the graph of f where you fix every input variable except for the ith one. Describe this new vertical slice, again, in terms of the size of this slice, where it resides and how it is situated with regard to the graph of f.

(definitions, viewpoint, visualization of functions)

Question 3. Prove (explain as precisely as you are able) the following:

(a) For $\mathbf{x} \in \mathbb{R}^n$, let $B_s(\mathbf{x})$ be the ball of radius s centered around the point $\mathbf{x}$. Show that for $s < t$, $B_s(\mathbf{x}) \subset B_t(\mathbf{x})$.

(b) The boundary points of the open interval $(a, b) \subset \mathbb{R}$ are precisely the points $x = a$ and $x = b$.

(sets, neighborhoods, boundary points)

Question 4. Show the following:

(a) The limit of $h(x, y) = \dfrac{4y^2 - x^2}{x^2 + y^2}$ at the origin in $\mathbb{R}^2$ does not exist.

(b) The function $i(x, y) = \begin{cases} \frac{xy^3}{x^2+y^6} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ is not continuous.

Question 5. Do the following. (The first two parts, in black, will be graded. The remaining three parts, in blue, are more challenging and will not be graded. Try them, if you care to):

(a) Use the $\varepsilon\delta$-definition of a limit (also see Example 4 and/or Example 12 in Section 2.2 of the text) to show that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0$.

(b) Use **part(a)** and some limit rules to show that , for $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy$, $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = x_0 y_0$.

(c) Calculate $\displaystyle\lim_{\mathbf{x}\to\mathbf{0}} \frac{\sin(xy)}{xy}$. (Hint: Write the function $\frac{\sin(xy)}{xy}$ as a composition of $\frac{\sin x}{x}$ and f and use the limit rules and your knowledge from Calculus I about the limit of the function $\frac{\sin x}{x}$ at $x = 0$.)

(d) For $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$, calculate $\displaystyle\lim_{\mathbf{x}\to\mathbf{0}} g(\mathbf{x})$, where $g(x,y,z) = \frac{\sin(xyz)}{xyz}$.

(e) Describe where $g(\mathbf{x})$ is continuous. Explain whether it is possible to make $g(\mathbf{x})$ continous by suitably defining it at $\mathbf{x} = \mathbf{0}$.

(limits, limit rules, continuity)