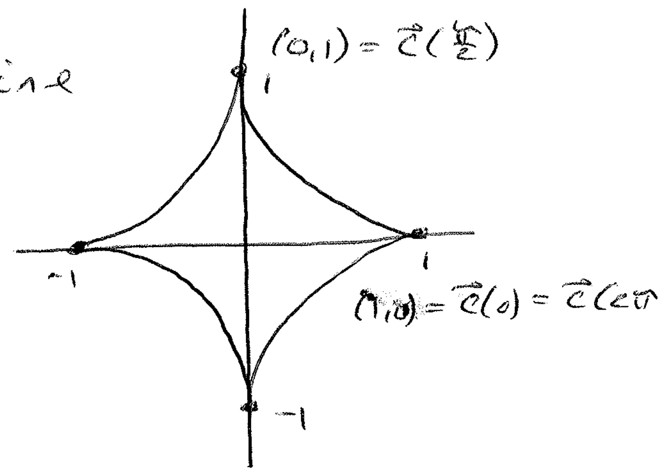
Problem 7.2.11 The image of the path

$t \longmapsto (\cos^3 t, \sin^3 t)$, $0 \le t \le 2\pi$ is shown

Evaluate the vector line integral of $\vec{F} = x\vec{\imath} + y\vec{\jmath}$ around the curve.



Strategy: We simply calculate the constituent parts of $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ and evaluate.

Solution: Here $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$, where

$\vec{F} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{c}(t) = \begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix}$, $\vec{c}\,'(t) = \begin{bmatrix} -3\cos^2 t \sin t \\ 3\sin^2 t \cos t \end{bmatrix}$, and

$\vec{F}(\vec{c}(t)) = \begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix}$. Thus

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \left(\begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix} \cdot \begin{bmatrix} -3\cos^2 t \sin t \\ 3\sin^2 t \cos t \end{bmatrix} \right) dt$$

$$= \int_0^{2\pi} -3\cos^5 t \sin t \, dt + \int_0^{2\pi} 3\sin^5 t \cos t \, dt$$

Problem 7.2.11 (cont'd).

$$\int \vec{F}\cdot d\vec{s} \ \underset{\substack{v=\sin t\\ dv=\cos t\,dt}}{\overset{\substack{u=\cos t\\ du=-\sin t\,dt}}{=}} \int_0^{\pi}(-3\cos^5 t\sin t)\,dt + \int_{\pi}^{2\pi}(-3\cos^5 t\sin t)\,dt$$

$$+\int_0^{\frac{\pi}{2}} 3\sin^5 t\cos t\,dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3\sin^5 t\cos t\,dt + \int_{\frac{3\pi}{2}}^{2\pi} 3\sin^5 t\cos t\,dt$$

(Breaking these integrals up is needed to render the substitutions one-to-one mappings).

$$=\int_1^{-1}(-3u^6)\,du + \int_{-1}^{1}(-3u^6)\,du$$

$$+\int_0^1 3v^6\,dv + \int_1^{-1} 3v^6\,dv + \int_{-1}^{0} 3v^6\,dv = 0$$

Another way to do this problem:

Since $\vec{F}=\begin{bmatrix} x\\ y\end{bmatrix} = \nabla f$, for $f(x,y)=\frac{1}{2}(x^2+y^2)$. we have

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \nabla f\cdot d\vec{s} = f(\vec{c}(2\pi)) - f(\vec{c}(0))$$

But since $\vec{c}(2\pi)=\vec{c}(0)$, we have

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = 0.$$

Problem 7.1.20 Show that the path integral of $f(x,y)$ over a path C given by the graph of $y = g(x)$, $a \le x \le b$ is:

$$\int_C f\,ds = \int_a^b f(x, g(x))\sqrt{1 + [g'(x)]^2}\,dx.$$

Conclude that if $g : [a,b] \to \mathbb{R}$ is piecewise differentiable, then the length of the graph of g on $[a,b]$ is:

$$\int_C f\,ds = \int_a^b \sqrt{1 + (g'(x))^2}\,dx$$

Solution: For $\vec{c} = \begin{bmatrix} x(t) \\ g(t) \end{bmatrix}$ a parameterization of C, we have

$$\int_{\vec{c}} f\,ds = \int_{\vec{c}} f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

$$= \int_{\vec{c}} f(x(t), g(t))\sqrt{(x'(t))^2 + (g'(t))^2}\,dt$$

Problem 7.1.20 (cont'd).

So here, we parameterize C directly using the independent variable x in $y=g(x)$.

Then $\vec{c}(t)=\vec{c}(x)=\begin{bmatrix} x \\ g(x) \end{bmatrix}$. Our integral is then

$$\int_C f\,ds = \int_{\vec{c}} f(\vec{c}(x))\,\|\vec{c}\,'(x)\|\,dx$$

$$= \int_a^b f(x, g(x))\sqrt{\left(\tfrac{d}{dx}(x)\right)^2 + \left(\tfrac{d}{dx}g(x)\right)^2}\,dx$$

$$= \int_a^b f(x, g(x))\sqrt{1+(g'(x))^2}\,dx \quad \text{as stated.}$$

To calculate the length of a piecewise differentiable curve, we calculate the path integral of the function $f(x,y)=1$ over $\vec{c}$:

$$\int_C f\,ds = \int_{\vec{c}} f(x,g(x))\sqrt{1+(g'(x))^2}\,dx = \int_a^b 1\,\sqrt{1+(g'(x))^2}\,dx$$

$$= \int_a^b \sqrt{1+(g'(x))^2}\,dx.$$