# 4.4.37

Let $\vec{F}(x,y,z) = 2xz^2\,\vec{i} + \vec{j} + xy^3z\,\vec{k}$

and $f(x,y,z) = x^2 y$

a) By definition, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Thus, $\nabla f = (2xy, x^2, 0) = 2xy\,\vec{i} + x^2\,\vec{j}$

b) Note that $\nabla \times \vec{F}$ is another name for curl $\vec{F}$

By definition, we have

$$\nabla \times F = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & 1 & xy^3z \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xy^3z) - \frac{\partial}{\partial z}(1)\right)\vec{i} - \left(\frac{\partial}{\partial x}(xy^3z) - \frac{\partial}{\partial z}(2xz^2)\right)\vec{j} + \left(\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(2xz^2)\right)\vec{k}$$

$$= 3xy^2z\,\vec{i} + (-y^3z + 4xz)\,\vec{j} \left(+\ 0\cdot\vec{k}\right)$$

$$= (3xy^2, -y^3z + 4xz, 0)$$

c) We already know $\vec{F}$ (given) and $\nabla f$ (computed in (a))

By definition of the cross product,

$$\vec{F} \times (\nabla f) = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2xz^2 & 1 & xy^3z \\ 2xy & x^2 & 0 \end{pmatrix}$$

$$= (1\cdot 0 - x^2\cdot xy^3z)\,\vec{i} - (2xz^2\cdot 0 - xy^3z\cdot 2xy)\,\vec{j} + (2xz^2\cdot x^2 - 1\cdot 2xy)\,\vec{k}$$

$$= -x^3y^3z\,\vec{i} + 2x^2y^4z\,\vec{j} + (2x^3z^2 - 2xy)\,\vec{k}$$

$$= (-x^3y^3z, 2x^2y^4z, 2x^3z^2 - 2xy)$$

d) Note that $F \cdot (\nabla f)$ should be scalar valued.

We have

$$\vec{F} \cdot (\nabla f) = (2xz^2, 1, xy^3z) \cdot (2xy, x^2, 0)$$

$$= 2xz^2 \cdot 2xy + 1 \cdot x^2 + xy^3z \cdot 0$$

$$= 4x^2yz^2 + x^2$$

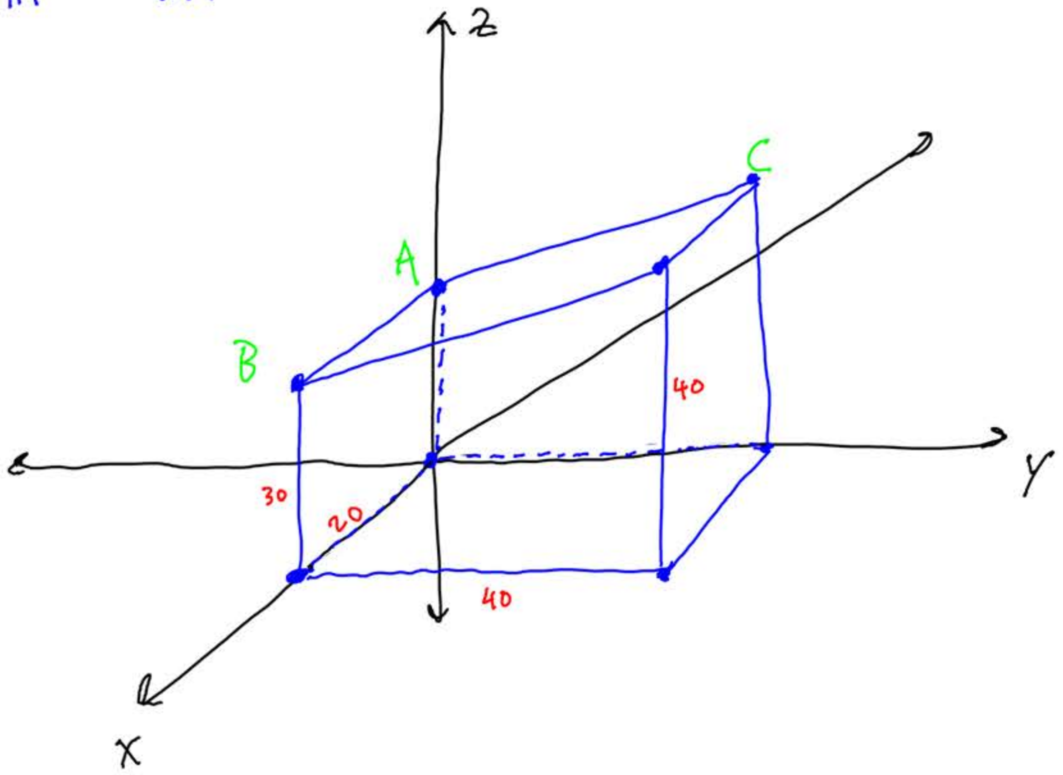
so it is indeed a scalar and not a vector.

$\blacksquare$

# 5.3.7

Consider a barn with a 20 ft × 40 ft rectangular base, vertical walls 30 ft high at the front (on the 20 ft side) and 40 ft high at the rear.

Let's start by drawing a picture. Since we're going to set up a double integral, we place the barn in in $\mathbb{R}^3$ with one of its corners at the origin




There are several ways to compute the volume of the barn as a double integral, but one way consists of viewing the roof of the barn as the graph of a function $f(x,y)$ over the rectangle $R$ at the base. The volume will then be given by

$$\iint_R f(x,y)\, dA.$$

First, we describe $R$ to get the limits of integration. It is given by

$$\begin{cases} 0 \le x \le 20 \\ 0 \le y \le 40 \end{cases} \quad R$$

Now we need to describe the roof of the barn as the graph of a function $f(x,y)$. It is a plane, so we start by finding its equation. It contains the points $A = (0,0,30)$, $B = (20,0,30)$, and $C = (0,40,40)$.

Then
$$\vec{AB} = (20,0,30) - (0,0,30) = (20,0,0)$$
and
$$\vec{AC} = (0,40,40) - (0,0,30) = (0,40,10)$$

Then a normal vector to the plane is given by
$$\vec{n} = \vec{AB} \times \vec{AC} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 20 & 0 & 0 \\ 0 & 40 & 10 \end{pmatrix} = 0\vec{i} - 200\vec{j} + 800\vec{k}$$

Using $A$ as our point on the plane, an equation for the plane is given by
$$\vec{n} \cdot [(x,y,z) - A] = 0$$
i.e.
$$(0,-200,800)\cdot(x,y,z-30) = 0$$

or

$$-200y + 800(z-30) = 0$$

We may solve for $z$:

$$z = \frac{1}{4}y + 30$$

Then setting $f(x,y) = \frac{1}{4}y + 30$, we have that the graph of $f$ describes our plane

We have reduced the problem of calculating the volume to computing the double integral

$$\iint_R f(x,y)\,dA = \int_0^{20}\int_0^{40} \frac{1}{4}y + 30 \,dy\,dx$$

$$= \int_0^{20} \frac{y^2}{8} + 30y \Big|_{y=0}^{y=40} dx$$

$$= \int_0^{20} \frac{40^2}{8} + 30\cdot 40 \,dx$$

$$= 20\left(\frac{40^2}{8} + 30\cdot 40\right)$$

$$= 28000$$

Thus, the volume of the barn is $28000 \text{ ft}^3$.