

3.4.6 In exercises 3 to 7, find the extrema of f subject to the stated constraints.

6. $f(x, y, z) = x + y + z$, subject to $x^2 - y^2 = 1, 2x + z = 1$

We use the content of the section **Several Constraints** (page 191), which states:

If a surface S is defined by a number of constraints, namely,

$$\left. \begin{aligned} g_1(x_1, \dots, x_n) &= c_1 \\ g_2(x_1, \dots, x_n) &= c_2 \\ &\vdots \\ g_k(x_1, \dots, x_n) &= c_k \end{aligned} \right\},$$

then the Lagrange multiplier theorem may be generalized as follows: If f has a maximum or minimum at $\mathbf{x}_0$ on S , there must exist constants $\lambda_1, \dots, \lambda_k$ such that

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \dots + \lambda_k \nabla g_k(\mathbf{x}_0)$$

This presents an interesting extension to the standard Lagrange procedure. Given any particular constraint g_i , the gradient vector ∇g_i points perpendicularly to the “surface” described by $g_i(x_1, \dots, x_n) = c_i$. Now, the multiple constraints $g_1, \dots, g_k$, describe a set consisting of the intersection of each of the individual surfaces $g_i = c_i$, and their gradient vectors $\nabla g_1, \dots, \nabla g_k$ (assuming that they’re linearly independent (what if they aren’t?)) all point perpendicularly to this resulting intersection surface, S . Their linear combinations form a linear subspace, and we might understand this space as the “orthogonal space” to S at any given point $\mathbf{x}_0$. The extended Lagrange procedure, then, commands us to find an $\mathbf{x}_0$ in S such that $\mathbf{x}_0$ ’s gradient $\nabla f(\mathbf{x}_0)$ lies in the constraint set S ’s orthogonal space. Such an $\mathbf{x}_0$, of course, will be an extremum on S .

Applying the above statement to our particular case, we see that we must find $\mathbf{x}_0 = (x, y, z)$ such that, for some constants λ_1, λ_2 , we have

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z)$$

$$g_1(x, y, z) = c_1, g_2(x, y, z) = c_2$$

or in particular,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2x \\ -2y \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$x^2 - y^2 = 1, 2x + z = 1$$

which we can also write as

$$\left. \begin{aligned} 1 &= \lambda_1 \cdot 2x + \lambda_2 \cdot 2 \\ 1 &= \lambda_1 \cdot -2y \\ 1 &= \lambda_2 \\ x^2 - y^2 &= 1, 2x + z = 1 \end{aligned} \right\}$$

The third equation gives $\lambda_2 = 1$. Substituting this into the first equation, we get $1 = \lambda_1 \cdot 2x + 1 \cdot 2 \rightarrow \lambda_1 \cdot 2x = -1$, while the second equation gives $\lambda_1 \cdot -2y = 1$. Therefore $\lambda_1 = -\frac{1}{2x} = -\frac{1}{2y} \rightarrow x = y$. However, the constraint $x^2 - y^2 = 1$ is never satisfied: there are no points (k, k) , for some number k , such that $(k)^2 - (k)^2 = 0 = 1$. Therefore, f has no extrema subject to the given constraints.

This makes sense. The constraint $x^2 - y^2 = 1$ describes a hyperbola which asymptotically approaches the diagonal lines $x - y = 0$ and $x + y = 0$. As this set marches upward and to the right within the xy -plane, for example, the value of f will keep climbing indefinitely. Meanwhile, as it marches upwards and to the left, the value of f will get more and more negative. This phenomenon is homogeneous as one travels along a branch of the hyperbola (say parameterized by $t = x$) and there are no local extrema on the way. Thus, there are no extrema in general.

3.4.17 Find the maximum and minimum values attained by $f(x, y, z) = xyz$ on the unit ball $x^2 + y^2 + z^2 \leq 1$.

Unlike that of the last problem, this constraint set is a compact domain (just wait till you study analysis!) so this function will actually attain extrema. Let's begin.

We cite the strategy outlined on page 194:

Lagrange Multiplier Strategy for Finding Absolute Maxima and Minima on Regions with Boundary Let f be a differentiable function on a closed and bounded region $D = U \cup \delta U$, U open in $\mathbb{R}^n$, with smooth boundary δU .

To find the absolute maximum and minimum of f on D :

- (i) Locate all critical points of f in U .
- (ii) Use the method of Lagrange multiplier to locate all the critical points of $f|_{\delta U}$.
- (iii) Compute the values of f at these critical points.
- (iv) Select the largest and smallest.

We carry out the procedure.

First, we find the critical points of f in U . $\nabla f = 0$ gives

$$\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow \begin{cases} yz = 0 \\ xz = 0 \\ xy = 0 \end{cases} \leftrightarrow \begin{cases} y = 0 \text{ OR } z = 0 \\ x = 0 \text{ OR } z = 0 \\ x = 0 \text{ OR } y = 0 \end{cases}$$

A bit of mental Boolean algebra tells us that the three equations on the right hold if and only if at least two out of three of x, y, z are equal to zero. So, we have critical points in U of the form $(x, 0, 0)$, $-1 < x < 1$; $(0, y, 0)$, $-1 < y < 1$; $(0, 0, z)$, $-1 < z < 1$. We'll set these aside for now and move on to the boundary.

Using the standard one-constraint Lagrange method on the boundary $x^2 + y^2 + z^2 = 1$, we get:

$$\begin{cases} \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, x^2 + y^2 + z^2 = 1 \leftrightarrow \begin{cases} yz = \lambda \cdot 2x \\ xz = \lambda \cdot 2y \\ xy = \lambda \cdot 2z \\ x^2 + y^2 + z^2 = 1 \end{cases} \end{cases}$$

This gives $\lambda = \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$. Cross-multiplying all possible pairs of equalities among the right-most three quantities, we get three expressions, $x^2 = y^2$, $y^2 = z^2$, $x^2 = z^2$, and transitivity gives $x^2 = y^2 = z^2$. This equation is satisfied on the unit sphere only on those points $(\pm k, \pm k, \pm k)$ such that $(\pm k)^2 + (\pm k)^2 + (\pm k)^2 = 1 \leftrightarrow 3k^2 = 1 \leftrightarrow k = \frac{1}{\sqrt{3}}$. Therefore, we get 8 critical points on the sphere: $(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$.

Now, we've assembled all of our critical points, and we compute their values.

$$\left. \begin{aligned} f(x, 0, 0) &= 0, \text{ for all } -1 < x < 1 \\ f(0, y, 0) &= 0, \text{ for all } -1 < y < 1 \\ f(0, 0, z) &= 0, \text{ for all } -1 < z < 1 \end{aligned} \right\}$$

$$f\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right) = \begin{cases} \frac{1}{3\sqrt{3}} & \text{if the number of positives is even} \\ -\frac{1}{3\sqrt{3}} & \text{if the number of positives is odd} \end{cases}$$

The critical points on the interior U all attain value zero, which is neither an absolute minimum nor an absolute maximum, so we can disregard them. The eight remaining critical points each achieve absolute maxima or minima, $\pm \frac{1}{3\sqrt{3}}$, respectively.

Whether the individual critical points are themselves absolute maxima or minima depends on the signs of their components. Critical points with an even number of positive components – $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ – attain the absolute maximum value, $\frac{1}{3\sqrt{3}}$. Critical points with an odd number of positive components $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ – attain the absolute minimum value, $-\frac{1}{3\sqrt{3}}$.