problem 3.2.9

Let's first recall that the general formula for the second Taylor expansion of function $f(x, y)$ at point (a, b) is

$$f(a,b)+\nabla f(a,b)\cdot(x-a,y-b)+\frac{1}{2}(\partial_{xx}f)(a,b)(x-a)^2+2(\partial_{xy}f)(a,b)(x-a)(y-b)+(\partial_{yy}f)(a,b)(y-b)^2$$

In this problem, we have $(a, b) = (\pi, \frac{\pi}{2})$, $f(x, y) = \cos x \sin y$ and so that

$$f(\pi, \frac{\pi}{2}) = -1 \tag{1}$$

$$\nabla f(\pi, \frac{\pi}{2}) = (-\sin(\pi)\sin(\frac{\pi}{2}), \cos(\pi)\cos(\frac{\pi}{2})) = (0, 0) \tag{2}$$

$$(\partial_{xx}f)(\pi, \frac{\pi}{2}) = -\cos(\pi)\sin(\frac{\pi}{2}) = 1 \tag{3}$$

$$(\partial_{xy}f)(\pi, \frac{\pi}{2}) = -\sin(\pi)\cos(\frac{\pi}{2}) = 0 \tag{4}$$

$$(\partial_{yy}f)(\pi, \frac{\pi}{2}) = -\cos(\pi)sin(\frac{\pi}{2}) = 1 \tag{5}$$

By plugging in the above values, the second order Taylor yields

$$-1 + \frac{1}{2}(x - \pi)^2 + \frac{1}{2}(y - \frac{\pi}{2})^2$$

problem 3.3.21

First let's compute the critical points of z we have

$$\nabla f(x, y) = (2xe^{1-x^2-y^2} - 2x(x^2 + 3y)e^{1-x^2-y^2}, 6ye^{1-x^2-y^2} - 2y(x^2 + 3y)e^{1-x^2-y^2})$$

so the critical points are obtained by solving the system

$$\begin{cases} 2xe^{1-x^2-y^2} - 2x(x^2 + 3y)e^{1-x^2-y^2} = 0 \\ 6ye^{1-x^2-y^2} - 2y(x^2 + 3y)e^{1-x^2-y^2} \end{cases}$$

Since exponential functions are always positive, the above system is equivalent to

$$\begin{cases} 2x - 2x(x^2 + 3y) = 0 \\ 6y - 2y(x^2 + 3y) = 0 \end{cases}$$ which admits 3 pairs of solutions $(0, 1)$,$(1, 0)$ and $(-1, 0)$.

Now let's determine the nature of those critical points, by computing $\partial_{xx}f(0, 1) = -4$, $\partial_{xx}f(1, 0) = \partial_{xx}f(-1, 0) = -12$ as well as $\partial_{xy}f(0, 1) = 0, \partial_{xy}f(1, 0) = -6$, $\partial_{xy}f(-1, 0) = 6$, and $\partial_{yy}f(0, 1) = -6$,$\partial_{yy}f(1, 0) = \partial_{yy}f(-1, 0) = 4$, we can compute the discriminate of each critical points,i.e., $D(0, 1) = \partial_{xx}f(0, 1)\partial_{yy}f(0, 1) - \partial_{xy}f(0, 1) = 24$, $D(1, 0) = -84$ and $D(-1, 0) = -84$.Therefore, we have immediate conclusion that $(1, 0)$ and $(-1, 0)$ are saddles, and since $\partial_{xx}f(0, 1) = -4 < 0$ and $D(0, 1) = 24 > 0$, one concludes $(0, 1)$ is the local max.