110.202: Week 5 Selected Problems

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Problem 2.5.11

Let

$$\mathbf{f}(x,y,z) = \begin{bmatrix} 3y+2 \\ x^2+y2 \\ x+z^2 \end{bmatrix} \quad \text{and} \quad \mathbf{c}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$$

(a) Find the path $\mathbf{p}(t) = (\mathbf{f} \circ \mathbf{c})(t)$, and the velocity vector $\mathbf{p}'(\pi)$. (b) Find $\mathbf{c}(\pi)$, $\mathbf{c}'(\pi)$, and $D\mathbf{f}(-1,0,\pi)$. (c) Thinking of $D\mathbf{f}(-1,0,\pi)$ as a linear map, find $(D\mathbf{f}(-1,0,\pi))\,\mathbf{c}'(\pi)$.

(a) We obtain $\mathbf{p}(t)$ by direct substitution:

$$\begin{aligned} \mathbf{p}(t) &= \mathbf{f}(\mathbf{c}(t)) \\ &= \begin{bmatrix} 3\sin t+2 \\ \cos^2 t+\sin^2 t \\ \cos t+t^2 \end{bmatrix} = \begin{bmatrix} 3\sin t+2 \\ 1 \\ \cos t+t^2 \end{bmatrix} \end{aligned}$$

We can differentiate $\mathbf{p}(t)$ componentwise with respect to t to obtain

$$\mathbf{p}'(t) = \begin{bmatrix} 3\cos t \\ 0 \\ -\sin t+2t \end{bmatrix}$$

Now, plugging in $t = \pi$, we find

$$\mathbf{p}'(\pi) = \begin{bmatrix} 3\cos\pi \\ 0 \\ -\sin\pi+2\pi \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2\pi \end{bmatrix}$$

(b) Plugging in $t = \pi$ in the given formula for $\mathbf{c}(t)$, we find that

$$\mathbf{c}(\pi) = \begin{bmatrix} \cos\pi \\ \sin\pi \\ \pi \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \pi \end{bmatrix}$$

We can differentiate $\mathbf{c}(t)$ componentwise with respect to t to obtain

$$\mathbf{c}'(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$$

Hence, by direct substitution,

$$\mathbf{c}'(\pi) = \begin{bmatrix} -\sin\pi \\ \cos\pi \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Recall that $D\mathbf{f}$ is the matrix of first partial derivatives of $\mathbf{f}$, where each *row* is an *output function* of $\mathbf{f}$, while each *column* is an *input variable*. Hence

$$D\mathbf{f}(x,y,z) = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 2x & 2y & 0 \\ 1 & 0 & 2z \end{bmatrix}$$

Plugging in the desired point, we find that

$$D\mathbf{f}(-1,0,\pi) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix}$$

(c) Using the values obtained above, we find that

$$(D\mathbf{f}(-1,0,\pi))(\mathbf{c}'(\pi)) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2\pi \end{bmatrix}$$

which is precisely $\mathbf{p}'(\pi)$! This result arises from the chain rule:

$$\mathbf{p}'(\pi) = D(\mathbf{f} \circ \mathbf{c})(\pi) = [D\mathbf{f}(\mathbf{c}(\pi))][D\mathbf{c}(\pi)] = (D\mathbf{f}(-1,0,\pi))(\mathbf{c}'(\pi))$$

Problem 2.6.4

You are walking along the graph of

$$f(x,y) = y\cos(\pi x) - x\cos(\pi y) + 10$$

standing at the point $(2,1,13)$. Find an x,y-direction you should walk in to stay at the same elevation.

This question is asking us to find a direction parallel to the level curve of f at the given point. There are many equivalent ways of solving this problem, but the first that comes to mind is to utilize one of the important facts we know about gradients, namely that the gradient of a function at a point is always *orthogonal* to the level curve of the function at that point.

In this case, some computation gives us that

$$\nabla f(x,y) := \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -\pi y\sin(\pi x) - \cos(\pi y) \\ \cos(\pi x) + \pi x \sin(\pi y) \end{bmatrix}$$

Plugging in the given x,y-point $(2,1)$, we find

$$\nabla f(2,1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now, we know that this vector is orthogonal to the level curve at $(2,1)$; hence, our desired direction $\mathbf{v} = \begin{bmatrix} v_x & v_y \end{bmatrix}^T$ must satisfy

$$\nabla f(2,1) \cdot \mathbf{v} = 0$$

Evaluating this dot product, we get that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} = v_x + v_y = 0$$

Hence, $v_y = -v_x$, and so our desired direction is given by

$$\mathbf{v} = \begin{bmatrix} v_x \\ -v_x \end{bmatrix} = v_x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for some arbitrary constant $v_x \in \mathbb{R}$.