

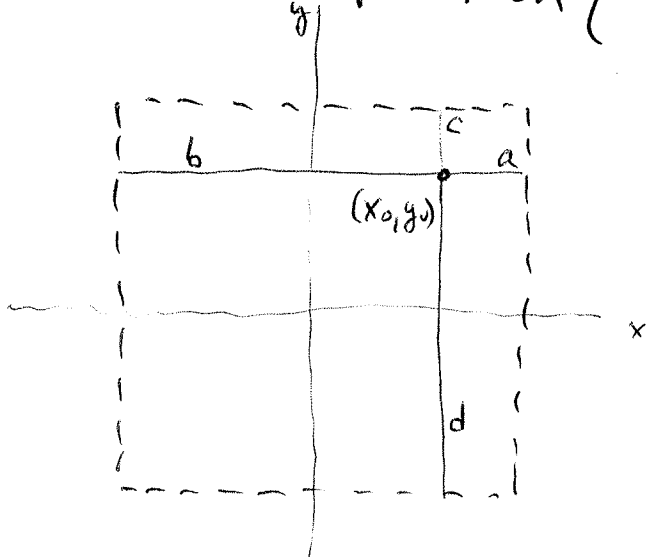
Problem 2.2.18

Let $A = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\}$. Show that A is an open subset of $\mathbb{R}^2$.

Strategy: We need to show that for any pt $(x_0, y_0) \in A$, there is an $\varepsilon > 0$ where $B_\varepsilon(x_0, y_0)$ lies completely inside A . We do this by construction.

Solution: Choose any $(x_0, y_0) \in A$, so that $-1 < x_0 < 1, -1 < y_0 < 1$. Define

$$\Gamma = \min \{ 1 - x_0, x_0 - (-1), 1 - y_0, y_0 - (-1) \}.$$



These are the 4 distances along lines parallel to the axes between (x_0, y_0) and the 4 boundary lines of A , labeled a, b, c, d , respectively.

and r is defined here as the smallest of these 4 distances.

Note 2 things:

- ① The distance between any pt in A and the boundary of A will always be ~~at least~~ the minimum of the 4 distances along these directions parallel to the axes. The triangle inequality is one way to see this. There are others. (see below)
- ② This is also true for any open ball of radius less than r centered at (x_0, y_0) , that its distance to the boundary of A will be positive and the minimum of the distances measured in the vertical and horizontal directions.

You can actually prove this using techniques in SVC. I won't ask you to, though.

② We also know $r \leq x_0 - (-1) = x_0 + 1$, where $x_0 + 1$ is the distance to the ~~right~~ left edge of the box. Hence

$$r \leq x_0 - (-1) = x_0 + 1 = x_0 - x + x + 1 \leq |x_0 - x| + x + 1$$

(same reason as before)

$$< \frac{1}{2}r + x + 1$$

$$\text{(since } |x_0 - x| < \varepsilon = \frac{1}{2}r \text{)}$$

But then $r \leq \frac{1}{2}r + x + 1$, or $\frac{1}{2}r < x + 1$ and since $\frac{1}{2}r$ is positive, $0 < x + 1$, or $\boxed{-1 < x}$

Hence by ① and ② $-1 < x < 1$.

③ and ④ are just like ① and ②, so that $-1 < y < 1$.

Thus if $(x, y) \in B_\varepsilon(x_0, y_0)$, where $\varepsilon = \frac{1}{2}r$, then $(x, y) \in A$.

Thus $B_\varepsilon(x_0, y_0) \subset A$ and we are done. $\square$