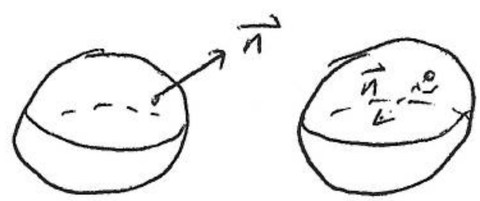
Lecture 36: ~~[illegible]~~

Recall that given $\vec{F}$ a C^1 vector field in $\mathbb{R}^3$ defined on an oriented surface, the quantity

$$\iint_S \vec{F} \cdot d\vec{S}$$

measures the flux of $\vec{F}$ through S (in the direction of the orientation (normal vectors)).

If the surface is closed (has no boundary) and bounded, then this integral measures the



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amount of fluid ~~flows~~ from the inside/outside if $\vec{n}$ points outside, from outside inside if $\vec{n}$ points in.

We have the following:

Thm (Gauss) Let D be a bounded solid region in $\mathbb{R}^3$ whose boundary ∂D consists of a finite number of piecewise smooth, closed orientable surfaces, each oriented by normals pointing away from D.

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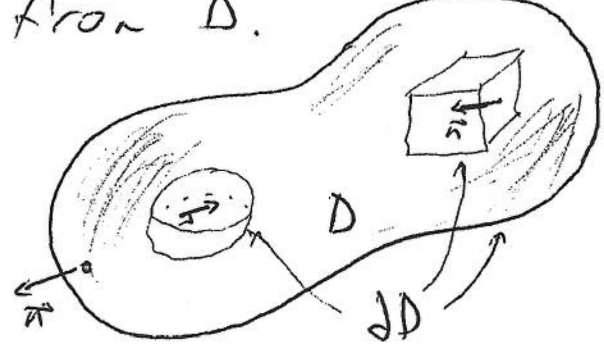
Then

$$\iint_{\partial D} \vec{F} \cdot d\vec{S} = \iiint_D (\nabla \cdot \vec{F})\, dV$$



Notes ① Left hand side is the flux of $\vec{F}$ across ∂D from inside D to outside D. We calculate for each piece and add together.

② Right hand side is the divergence of $\vec{F}$, $\text{div}(F) = \nabla \cdot \vec{F}$, a scalar valued function defined on D. This is a scalar triple integral.

③ If S is a closed surface, so $\partial S = \phi$, we sometimes write $\oiint_S F \cdot d\vec{S}$.

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Notes (cont'd)

③ (Cont'd.) Whenever D is a 3-dim region with boundary, $S = \partial D$, S has no boundary as a surface.

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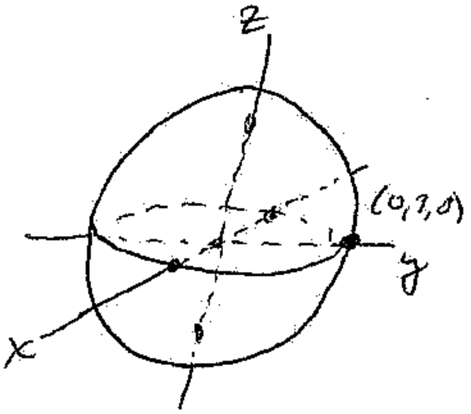
④ This gives us an interpretation for divergence:

One can measure the total (aggregate) divergence of a vector field inside a closed bounded domain in $\mathbb{R}^3$ by instead measuring the total (aggregate) flux of the vector field through the surface boundary (inside to out).

ex Verify Gauss Thm for $\vec{F} = 3x\vec{i} + 2y\vec{j}$ where B is the ball of radius 3 centered at the origin.



Strategy: Parameterize $S = \partial B$ so that normal points out. Then calculate each side of

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_B (\mathrm{div}\,\vec{F})\, dV.$$

Solution Here $\mathrm{div}(\vec{F}) = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(0) = 5.$

Hence $$\iiint_B (\mathrm{div}\,\vec{F})\, dV = \iiint_B 5\, dV = 5\iiint_B dV$$

$$= 5(\text{volume of } B)$$

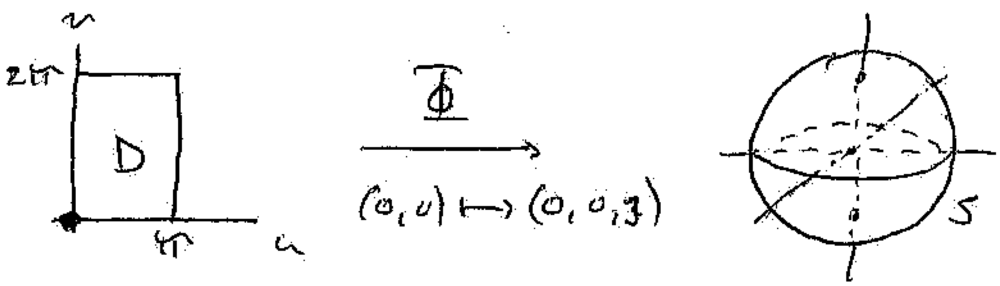
$$= 5\left(\tfrac{4}{3}\pi(3)^3\right)$$

$$= 180\pi.$$

ex. (cont'd) Solution (cont'd).

To calculate $\oiint_S \vec{F}\cdot d\vec{S}$, we parameterize $S = \partial B$

$$\Phi(u,v) = (3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u)$$



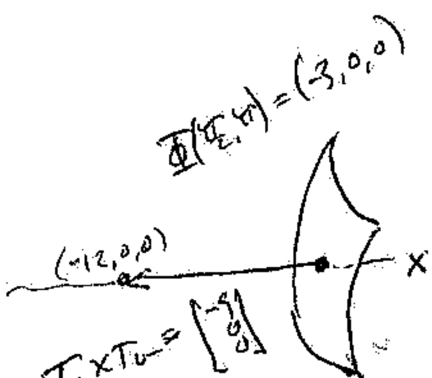
Then $\oiint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v))\cdot(\vec{T}_u\times\vec{T}_v)\,du\,dv$

- $\vec{F}(\Phi(u,v)) = \begin{bmatrix} 3(3\sin u\cos v) \\ 2(3\sin u\sin v) \\ 0(3\cos u) \end{bmatrix} = \begin{bmatrix} 9\sin u\cos v \\ 6\sin u\sin v \\ 0 \end{bmatrix}$

- $\vec{T}_u\times\vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos u\cos v & 3\cos u\sin v & -3\sin u \\ -3\sin u\sin v & 3\sin u\cos v & 0 \end{vmatrix}$

$$= (9\sin^2 u\cos v)\,\vec{i} - (-9\sin^2 u\sin v)\,\vec{j} + (9\cos^2 v\sin u\cos u + 9\sin^2 v\sin u\cos u)\,\vec{k}$$

$$= \begin{bmatrix} 9\sin^2 u\cos v \\ 9\sin^2 u\sin v \\ 9\sin u\cos u \end{bmatrix}$$



Here, at $u=\frac{\pi}{2}$, $v=\pi$, $\Phi(\frac{\pi}{2},\pi) = (-3,0,0)$ and $\vec{T}_u\times\vec{T}_v = \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}$

Facing outward.

ex.(cont'd.) Solution (cont'd).

So $\quad \vec{F}(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v) = \begin{bmatrix} 9\sin u \cos v \\ 6 \sin u \sin v \\ 0 \end{bmatrix} \begin{bmatrix} 9 \sin^2 u \cos v \\ 9 \sin^2 u \sin v \\ 9 \sin u \cos u \end{bmatrix}$

$$= 81 \sin^3 u \cos^2 v + 54 \sin^3 u \sin^2 v$$

$$= (27 \cos^2 v + 54(\cos^2 v + \sin^2 v)) \sin^3 u$$

$$= 27 \sin^3 u (\cos^2 v + 2)$$

And $\displaystyle \oiint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} 27(\cos^2 v + 2) \int_0^{\pi} \sin^3 u \, du \, dv$

$$= 27 \int_0^{2\pi} (\cos^2 v + 2) \underbrace{\int_0^{\pi} (1 - \cos^2 u) \sin u \, du}_{} \, dv$$

Let $x = \cos u$
$dx = -\sin u \, du$
when $u = 0$, $x = 1$
$u = \pi$, $x = -1$

$$= 27 \int_0^{2\pi} (\cos^2 v + 2) \int_1^{-1} (1 - x^2)(-dx) \, dv$$

$$= 27 \int_0^{2\pi} (\cos^2 v + 2) \left(x - \frac{x^3}{3} \right) \Big|_1^{-1} \, dv$$

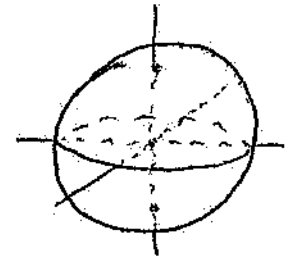
$$= 27 \int_0^{2\pi} (\cos^2 v + 2)\left(\frac{4}{3}\right) du = 36 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2v + 2 \right) dv$$

$$= 36 \left(\frac{5}{2} v + \frac{1}{4} \sin 2v \Big|_0^{2\pi} \right) = 36\left(\frac{5}{2}\right)(2\pi)$$

$$= 180\pi.$$

ex. Let $\vec{F} = (e^y \cos z)\vec{i} + (\sqrt{x^3+1}\sin z)\vec{j} + (x^2+y^2+3)\vec{k}$

Calculate $\oiint_S \vec{F}\cdot d\vec{S}$ where $S = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=9\}$.



Strategy: Use Gauss' Thm

Solution: Notice that

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(e^y\cos z) + \frac{\partial}{\partial y}(\sqrt{x^3+1}\sin z) + \frac{\partial}{\partial z}(x^2+y^2+3)\vec{k}$$

$$= 0.$$

Hence by Gauss' Thm,

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_B (\operatorname{div}\vec{F})\,dV = 0$$

where $B = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \le 9\}$. ∎

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ex. Let $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, and D be the upright cylinder of radius a and height b centered on the z-axis and sitting on the xy-plane ($z=0$). Verify Gauss' Thm.

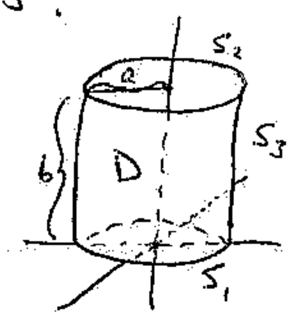
ex. cont'd.

Solution: First, we calculate $\iiint_D \operatorname{div} \vec{F}\, dV$.

Here $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$.

$$\text{So } \iiint_D \operatorname{div} \vec{F}\, dV = \iiint_D 3\, dV = 3 \iiint_D dV = 3(\text{vol } D)$$
$$= 3(\pi a^2 b).$$

We now show this is equal to $\oiint_{\partial D} \vec{F} \cdot d\vec{S}$.



Here ∂D has 3 pieces:

S_1: Think of $S_1 = \text{graph}(f)$, for $f(x,y) = 0$.

Then $\Phi(x,y) = (x,y,0)$ is a parameterization of S_1 in $\mathbb{R}^3$.

Then $\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$ points up. But this is the wrong orientation for Gauss' Thm.

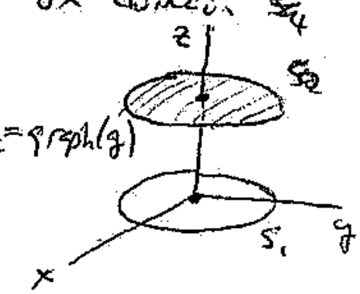
Hence reparameterize S_1 by $\Psi(x,y) = (-x,y,0)$

Then $\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -\vec{k}$ is pointing out of D.

Then $\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \begin{bmatrix} -x \\ y \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} dx\,dy = 0$. This makes sense as $\vec{F}$ is parallel to S_1 hence $\iint_{S_1} \vec{F} \cdot dS = 0$.

ex. (cont'd) Solution (cont'd).

S_2: Think of $S_2 = \text{graph}(g)$, $g(x,y) = b$, on domain
$S_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq a^2\}$.



Parameterize S_2 by

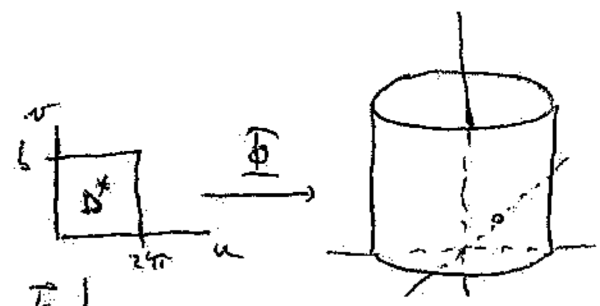
$$\Phi(x,y) = (x,y,b) \in \mathbb{R}^3.$$

Then $\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$ correctly pointing up. (outward from D).

Thus $\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \begin{bmatrix} x \\ y \\ b \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dx\,dy = \iint_{S_2} b \, dx\,dy$

$$= b \iint_{S_2} dx\,dy = b\,(\text{area } S_2) = b\pi a^2$$

S_3: Parameterize S_3 by
$\Phi(u,v) = (a\cos u, a\sin u, v)$



Then $\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos u)\vec{i} - (-a\sin u)\vec{j}$

Note that at $u=v=0$, $\Phi(0,0) = (a,0,0)$, and here $\vec{T}_u \times \vec{T}_v = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$. Hence normal points out of D and this is the correct orientation.

ex. (cont'd.) Solution (cont'd)

S_3: Then $\iint_{S_3} \vec{F} \cdot d\vec{S} = \iint_{D^*} \begin{bmatrix} a\cos u \\ a\sin u \\ v \end{bmatrix} \cdot \begin{bmatrix} a\cos u \\ a\sin u \\ 0 \end{bmatrix} du\,dv$

$$= \iint_{D^*} a^2\,du\,dv = a^2(\text{area } D_3^*) \cancel{\quad}.$$

$$= a^2(2\pi b) = 2\pi a^2 b.$$

And then

$$\oiint_{\partial D} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F} \cdot d\vec{S}$$

$$= 0 + \pi b a^2 + 2\pi a^2 b = 3\pi a^2 b. \blacksquare$$