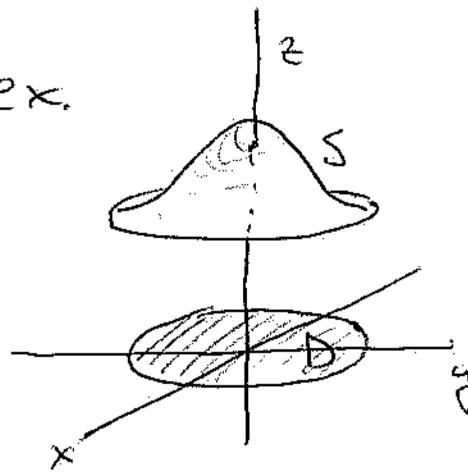
Lecture 35

We start with a creative application of Stokes' Theorem:

ex.



$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l}(x,y) \in D \\ z = e^{-(x^2+y^2)}\end{array} \right\}$$
$$= \left\{ (x,y,z) \in \mathbb{R}^3 \mid z \geq \tfrac{1}{e},\ z = e^{-(x^2+y^2)} \right\}.$$

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1 \}.$$

Calculate the flux of the curl of
$$\vec{F} = (e^{y+z} - 2y)\,\vec{i} + (xe^{y+z} + y)\,\vec{j} + (e^{x+y})\,\vec{k}$$
across S, the graph of $f: D \to \mathbb{R}$, $f(x,y) = e^{-(x^2+y^2)}$

Attempt 1: Direct calculation of $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

Here $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D \nabla \times \vec{F}(x,y,e^{-(x^2+y^2)}) \cdot \vec{N}(x,y,e^{-(x^2+y^2)})\,dx\,dy$

So we will need $\nabla \times \vec{F}$ and $\vec{N} = \vec{T}_x \times \vec{T}_y$, since (x,y) parameterizes S, for $\Phi: D \to \mathbb{R}^3$
$\Phi(x,y) = (x, y, e^{-(x^2+y^2)})$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y+z} - 2y & xe^{y+z} + y & e^{x+y} \end{vmatrix}$$

$$= (e^{x+y} - xe^{y+z})\vec{i} - (e^{x+y} - e^{y+z})\vec{j} + 2\vec{k}$$

and

$$\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2xe^{-(x^2+y^2)} \\ 0 & 1 & -2ye^{-(x^2+y^2)} \end{vmatrix}$$

$$= 2xe^{-(x^2+y^2)}\vec{i} + 2ye^{-(x^2+y^2)}\vec{j} + \vec{k}$$

Hence $$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left. \begin{bmatrix} e^{x+y} - xe^{y+z} \\ -e^{x+y} + e^{y+z} \\ 2 \end{bmatrix} \right|_{z = e^{-(x^2+y^2)}} \cdot \begin{bmatrix} 2xe^{-(x^2+y^2)} \\ 2ye^{-(x^2+y^2)} \\ 1 \end{bmatrix} dx\,dy.$$

$$= \ldots \text{ (This is a tough calculation)}$$

Attempt 2: Using Stokes' Thm directly

$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$. To do this, we parameterize $\partial S = \{(x, y, \frac{1}{e}) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ as $\partial S = \vec{c}$, with $\vec{c} : [0, 2\pi] \to \mathbb{R}^3$, $\vec{c}(t) = \begin{bmatrix} \cos t \\ \sin t \\ \frac{1}{e} \end{bmatrix}$

Here $\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\cos t, \sin t, \frac{1}{e}) \cdot \vec{c}\,'(t)\, dt$

$$= \int_0^{2\pi} \left.\begin{bmatrix} e^{y+z} - 2y \\ x e^{y+z} + y \\ e^{x+y} \end{bmatrix}\right|_{\substack{x=\cos t \\ y = \sin t \\ z = \frac{1}{e}}} \cdot \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix} dt$$

$$= \int_0^{2\pi} \left[\left(e^{\sin t + \frac{1}{e}} - 2\sin t\right)(-\sin t) + (\cos t) \left(e^{\sin t + \frac{1}{e}} + \sin t\right)(\cos t) \right] dt$$

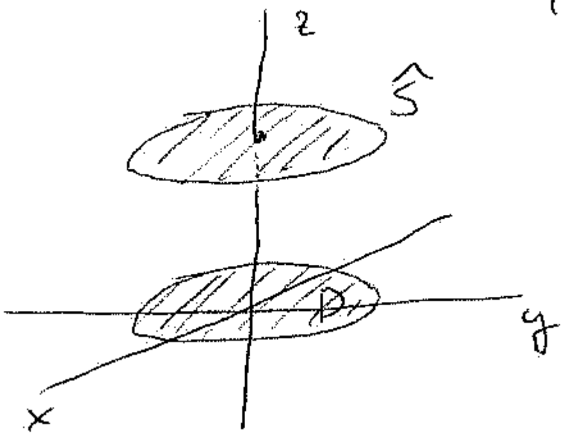
= also very hard calculation ....

Attempt 3: Use Stokes' Thm indirectly.

Create a new surface with same boundary as S:

Let $\hat{S} = \{(x, y, \frac{1}{e}) \in \mathbb{R}^3 \mid x^2 + y^2 \le 1\}$. Then

$\partial \hat{S} = \partial S = \{(x, y, \frac{1}{e}) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.



Here, $\hat{S}$ is the flat disk at height $z = \frac{1}{e}$.

By Stoke's Thm

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{\partial \hat{S}} \vec{F} \cdot d\vec{s} = \underbrace{\iint_{\hat{S}} (\nabla \times \vec{F}) \cdot d\vec{S}}$$

Here, we do the last calculation

To do this, we write $\iint_{\hat{S}} (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{\hat{S}} (\nabla \times \vec{F} \cdot \vec{n}) \, dS$,

noting that now $\vec{n} = \vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So $\nabla \times \vec{F} \cdot \vec{n} = \begin{bmatrix} e^{x+y} - xe^{y+z} \\ e^{y+z} - e^{x+y} \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 2$.

So $\iint_{\hat{S}} (\nabla \times \vec{F} \cdot \vec{n}) \, dS = \iint_{\hat{S}} 2 \, dS = 2 \iint_{\hat{S}} dS$

$= 2(\text{area } \hat{S}) = 2(\pi (1)^2) = 2\pi$.

Recall the definition of a conservative vector field:

<u>Def</u> A C^1-vector field $\vec{F}$ in $\mathbb{R}^n$ is <u>conservative</u> if $\vec{F} = \nabla f$, for $f: \mathbb{R}^n \to \mathbb{R}$.

In $\mathbb{R}^3$, we have the following:

<u>Thm</u> Let $\vec{F}$ be a C^1-vector field on $\mathbb{R}^3$. The following are equivalent:

(a) F is conservative.

(b) For any simple closed curve, $\int_C \vec{F} \cdot d\vec{s} = 0$.

(c) For any 2 oriented simple curves C_1 and C_2, oriented so that the beginning and end pts are the same,

$$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$$

(d) $\nabla \times \vec{F} = 0$.

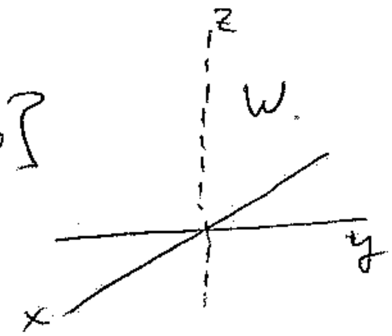
Hence gradient fields are irrotational, and irrotational vector fields are gradient fields.

But this only works here, <u>on all of $\mathbb{R}^3$</u>.

Example of an irrotational, <u>non</u> conservative vector field <u>in</u> $\mathbb{R}^3$:

Let $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + 0\vec{k}$. As a function, its domain is only

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \neq 0\}$$
$$= \mathbb{R}^3 - \{z\text{-axis}\}$$



Here, on W, $\vec{F}$ is irrotational, since

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

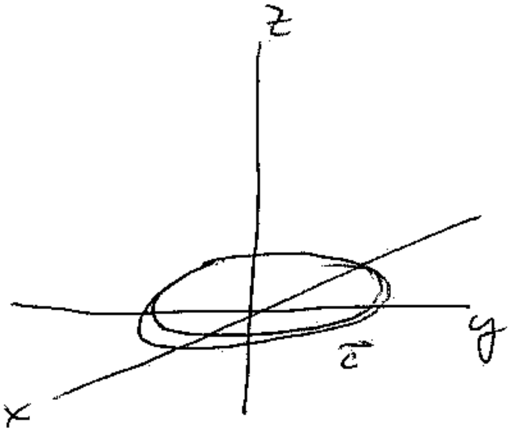
$$= 0\vec{i} - 0\vec{j} + \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\vec{k}$$

$$= 0\vec{i} - 0\vec{j} + \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)\vec{k}$$

$$= 0\vec{i} - 0\vec{j} + \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}\vec{k}$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$$

But F is not conservative on W. To see this, construct a single curve $\vec{c} \subset W$, where $\int_{\vec{c}} \vec{F} \cdot d\vec{s} \neq 0$.



Let $\vec{c}:[0,2\pi]\to\mathbb{R}^3$,
$\vec{c}(t)=\begin{bmatrix}\cos t\\ \sin t\\ 0\end{bmatrix}$ be the unit circle
in the xy-plane of xyz-space.

Here, $\vec{c}\in W$, and $\int_{\vec{c}}\vec{F}\bullet d\vec{r}=\int_0^{2\pi}\vec{F}(\cos t,\sin t,0)\bullet\vec{c}\,'(t)\,dt$.

where $\vec{F}(\vec{c}(t))=\begin{bmatrix}\frac{-\sin t}{\sin^2 t+\cos^2 t}\\ \frac{\cos t}{\sin^2 t+\cos^2 t}\\ 0\end{bmatrix}=\begin{bmatrix}-\sin t\\ \cos t\\ 0\end{bmatrix}=\vec{c}\,'(t)$.

So $\int_{\vec{c}}\vec{F}\bullet d\vec{r}=\int_0^{2\pi}\vec{F}(\vec{c}(t))\bullet\vec{c}\,'(t)\,dt=\int_0^{2\pi}\begin{bmatrix}-\sin t\\ \cos t\\ 0\end{bmatrix}\bullet\begin{bmatrix}-\sin t\\ \cos t\\ 0\end{bmatrix}dt$

$$=\int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt=\int_0^{2\pi}dt=t\Big|_0^{2\pi}=2\pi\neq 0.$$

So what is wrong here? The problem is that the thm only applies to vector fields defined on <u>all</u> of $\mathbb{R}^3$.
It also works more generally (when the domain of $\vec{F}$ is <u>simply connected</u>, which W is not).

For now, just be careful!

~~Notes (1) Hence gradient vector fields are irrotational and irrotational vector fields are gradient fields.~~

More examples

ex. Let $\vec{F} = (e^x \sin y - yz)\vec{i} + (e^x \cos y - xz)\vec{j} + (z - xy)\vec{k}$

Show $\vec{F}$ is conservative and find a potential f for it.

Strategy: We show it is irrotational and integrate to find f.

Solution: We calculate $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$, if $\operatorname{curl}(\vec{F}) = \vec{0}$ then by the above, f exists.

Here
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y - yz & e^x \cos y - xz & z - xy \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(z - xy) - \frac{\partial}{\partial z}(e^x \cos y - xz)\right)\vec{i} - \left(\frac{\partial}{\partial x}(z - xy) - \frac{\partial}{\partial z}(e^x \sin y - yz)\right)\vec{j} \;\ldots$$

$$= (-x + x)\vec{i} - (-y + y)\vec{j} + \ldots$$

$= \vec{0}$

Solution (cont'd)

Hence there is a function $f : \mathbb{R}^3 \to \mathbb{R}$ where $\nabla f = \vec{F}$. This means:

$$\begin{aligned} &① \quad \frac{\partial f}{\partial x}(x,y,z) = e^x \sin y - yz \\ &② \quad \frac{\partial f}{\partial y}(x,y,z) = e^x \cos y - xz \\ &③ \quad \frac{\partial f}{\partial z}(x,y,z) = z - xy. \end{aligned}$$

In essence find an antiderivative!

Choose the first and note that $f(x,y,z) = \int \frac{\partial f}{\partial x}(x,y,z)\,dx$.

So take ① and integrate

$$f(x,y,z) = \int (e^x \sin y - yz)\,dx = e^x \sin y - xyz + \underbrace{g(y,z)}$$

plays the role of C here.

where $g(y,z)$ is some function of only y,z. (differentiate both sides to see).

And since $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^x \sin y - xyz + g(y,z)\right)$

$$= e^x \cos y - xz + \frac{\partial g}{\partial y} \underset{\text{equal}}{\overset{\text{must}}{=}} \underbrace{e^x \cos y - xz}_{②}$$

we conclude that $\frac{\partial g}{\partial y} = 0$. Hence g is only a function of z.

Solution (cont'd.)

Hence $f(x,y,z) = e^x \sin y - xyz + g(z)$.

Take partial with respect to z and compare to ③.

$$\frac{\partial}{\partial z}\left(e^x \sin y - xyz + g(z)\right) = -xy + \frac{dg}{dz}.$$

This must equal $z - xy$. We conclude that

$$\frac{dg}{dz} = z, \text{ or } g(z) = \frac{z^2}{2} + C$$

Hence $f(x,y,z) = e^x \sin y - xyz + \frac{z^2}{2} + C$.

Some final notes

(I) Recall ① for $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$, that $\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$, so that the scalar curl of $\vec{F}$ is $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$.

Corollary if $\vec{F}$ is C^1 on $\mathbb{R}^2$ and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, then there exists A on $\mathbb{R}^2$ where $\nabla A = \vec{F}$.

ex. Show $\vec{F} = (2xy + \cos 2y)\vec{i} + (x^2 - 2x\sin 2y)\vec{j}$ is conservative, and find a potential f.

Strategy: Write $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ and use Corollary. Integrate to find f.

Solution: Here

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy + \cos 2y) = 2x - 2\sin 2y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 - 2x\sin 2y) = 2x - 2\sin 2y$$

are equal. Hence by corollary $\vec{F}$ is conservative.

To find f, integrate to get

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int (2xy + \cos 2y)\,dx$$

$$= x^2y + x\cos 2y + g(y)$$

where g is a function of y alone.

But then $\frac{\partial}{\partial y}(f(x,y)) = \frac{\partial}{\partial y}(x^2y + x\cos 2y + g(y))$

$$= x^2 - 2x\sin 2y + \frac{dg}{dy}$$

Solution (cont'd)

must equal $Q(x,y) = x^2 - 2x\sin 2y$. Hence

$\frac{\partial g}{\partial y} = 0$. Thus $g(y) = C$ a constant.

$$f(x,y) = x^2 y + x\cos 2y + C.$$

∎

(II) Recall that we also had $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$ for $\vec{F}$ a C^2 vector field on $\mathbb{R}^3$. We can use this:

Thm For $\vec{F}$ a C^1-vector field with $\operatorname{div}(\vec{F}) = 0$, there exists a C^2 vector field $\vec{G}$, where $\vec{F} = \operatorname{curl}(\vec{G}) = \nabla \times \vec{G}$.

Note: Finding $\vec{G}$ is not easy, though.