Lecture 34: [redacted]

Green's Thm relates the curl (twisting effect) of a vector field on a domain $D \subset \mathbb{R}^2$ to the line integral of $\vec{F}$ along ∂S.

But all of this happened in $\mathbb{R}^2$. Place D in $\mathbb{R}^3$ as a surface (with boundary) and the ideas should still hold.... They do, and the result is attributed to Sir George Gabriel Stokes:

<u>Stokes Thm</u> Let S be a bounded piecewise smooth oriented surface in $\mathbb{R}^3$, where ∂S is a finite set of piecewise C^1-closed curves each oriented consistently with respect to S.

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For $\vec{F}$ a C^1-vector field on S, we have

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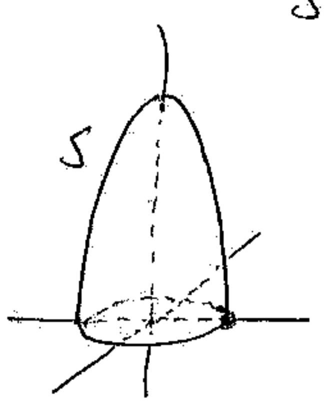
$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

<u>Note</u>: Again the aggregate twisting effect of $\vec{F}$ on S can be measured ~~by~~ along just the boundary via the vector line integral of $\vec{F}$ on ∂S.

ex. Let $f(x,y) = 9 - x^2 - y^2$ and define

$S = \text{graph}(f)$ on the domain $D = \{(x,y) \mid x^2+y^2 \le 9\}$



Here $\partial S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=9,\ z=0\}$.

Verify Stokes Thm for the vector field

$$\vec{F} = (2z-y)\vec{i} + (x+z)\vec{j} + (3x-2y)\vec{k}$$

<u>Strategy</u>: Calculate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ and $\int_{\partial S} \vec{F} \cdot d\vec{s}$ separately and verify equality.

<u>Solution</u>: First, we calculate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

Here S is parameterized by ~~[illegible]~~. (x,y) since it is the graph of $z = f(x,y) = 9 - x^2 - y^2$. and oriented as such.

Solution (cont'd.)

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D (\nabla \times \vec{F}) \cdot (\vec{T}_x \times \vec{T}_y)\, dx\, dy$$

In parts:

- $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z-y & x+z & 3x-2y \end{vmatrix} = (-2-1)\vec{i} - (3-2)\vec{j} + (1-(-1))\vec{k}$

$$= -3\vec{i} - \vec{j} + 2\vec{k}$$

- $\vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$

$$= 2x\vec{i} + 2y\vec{j} + \vec{k}$$

Then $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D \left(\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} \right) dx\, dy$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (-6x - 2y + 2)\, dy\, dx$$

$$= \int_{-3}^{3} \left(-6xy - y^2 + 2y \Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \right) dx$$

$$= \int_{-3}^{3} \Big(-6x\sqrt{9-x^2} - 9 + x^2 + 2\sqrt{9-x^2} - \left(-6x(-\sqrt{9-x^2}) - 9 + x^2 + 2(-\sqrt{9-x^2})\right) \Big) dx$$

$$= \int_{-3}^{3} -12x\sqrt{9-x^2}\, dx + \int_{-3}^{3} 4\sqrt{9-x^2}\, dx$$

Solution (cont'd)

Here the first integral is 0 since $-12x\sqrt{1-x^2}$ is an odd function. Thus

$x = 3\cos u$
$dx = -3\sin u\,du$
$x = 3 \quad u = 0$
$x = -3 \quad u = \pi$

$$\int_{\pi}^{0} 4\sqrt{9-9\cos^2 u}\,(-3\sin u)\,du = \int_0^{\pi} 36\sin^2 u\,du$$

$$= \int_0^{\pi} 36\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2u\right)du = 18\left(u - \tfrac{1}{2}\sin 2u\Big|_0^{\pi}\right)$$

$$= 18\pi.$$

Now we calculate $\oint_{\partial S} \vec{F}\cdot d\vec{s}$. Here $C = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=9\}$

$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_C \vec{F}(\vec{c}(t))\cdot(\vec{c}\,'(t))\,dt$ where we parametrize C by $\vec{c}: [0,2\pi]\to\mathbb{R}^3$, $\vec{c}(t) = (3\cos t, 3\sin t, 0)$. Then

$$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}(3\cos t, 3\sin t, 0)\cdot\begin{bmatrix}-3\sin t\\ 3\cos t\\ 0\end{bmatrix}dt = \int_0^{2\pi}\left(\begin{bmatrix}-3\sin t\\ 3\cos t\\ 9\cos t - 6\sin t\end{bmatrix}\cdot\begin{bmatrix}-3\sin t\\ 3\cos t\\ 0\end{bmatrix}\right)dt$$

$$= \int_0^{2\pi}(9\sin^2 t + 9\cos^2 t)\,dt = \int_0^{2\pi} 9\,dt = 9t\Big|_0^{2\pi} = 18\pi.$$

Q: Which way was easier?

Some notes

① Suppose the surface in $\mathbb{R}^3$ has no boundary (like S^2, or $\mathbb{T}^2$, etc). Then Stokes Thm says that

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \oint_{\cancel{\partial S}} \overset{0}{\cancel{\vec{F}\cdot d\vec{r}}} = 0.$$

The aggregate twisting effect of $\vec{F}$ on a surface with no boundary in $\mathbb{R}^3$ is 0.

② For a surface with boundary, the aggregate twisting effect of a vector field on a surface in $\mathbb{R}^3$ can be measured by its circulation along the boundary.

This is the same as Green's Thm, although both the surface and the vector field now reside in $\mathbb{R}^3$.

Recall that when $\vec{F} = \nabla f$, the line integral of $\vec{F}$ along a curve $\vec{c}:[a,b] \to \mathbb{R}^n$ was

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a)) \qquad (FTC)$$

The value of the integral depended only on the endpts of the curve.

And if the curve is closed, then $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = 0$!!!

Def. A C^1 vector field $\vec{F}$ in $\mathbb{R}^n$ is called conservative if $\vec{F} = \nabla f$ for $f: \mathbb{R}^n \to \mathbb{R}$.

Thm The Following are Equivalent: For $\vec{F}$ a C^1 vector field on $\mathbb{R}^3$

(a) $\vec{F}$ is conservative.

(b) For C a simple closed curve, $\int_C \vec{F} \cdot d\vec{s} = 0$

(c) For any 2 oriented simple curves ~~with~~ C_1 and C_2 with the same endpts,

$$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$$

(d) $\nabla \times \vec{F} = \vec{0}$.

Notes ① Hence gradient vector fields are irrotational, and irrotational vector fields are gradient fields, on $\mathbb{R}^3$

ex.