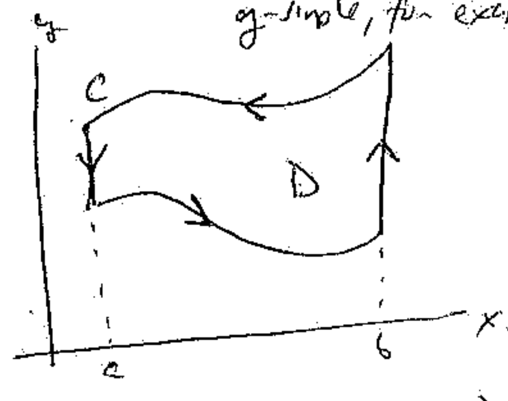
Lecture 33:

Given a closed, bounded region in $\mathbb{R}^2$, integrating a quantity in the region can sometimes be performed by instead integrating something related on its boundary. This replaces a regular double integral with a line integral on the edge (which is a closed curve).

Let D be an elementary region in $\mathbb{R}^2$, and let $\partial D = C$ be its boundary. Here, C can be oriented in 2 ways. Orient it counterclockwise

y-simple, for example.



(if you walk along C, D is always on your left).

Let $\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ be a C^1-vector field defined on D. We have the following:

Thm (Green) For D as above, with $\vec{F}$ as above, we have

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$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Notes ① The left hand side is

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \int_{\partial D} P(x,y)\,dx + Q(x,y)\,dy$$

② The vector line integral of $\vec{F}$ over $C = \partial D$ equals the scalar surface integral of $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$ over D.

③ In the case (like here) where $\partial D = C$ is a closed curve, we sometimes write

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \oint_{\partial D} \vec{F} \cdot d\vec{s} = \oint_C \vec{F} \cdot d\vec{s}$$

④ The idea of relating a 2-dim integral to a 1-dim integral is similar to the FTC

⑤ If ∂D consists of multiple closed curves, the theorem still holds:

Let D be closed and bounded, with $\partial D = C_1 + C_2 + C_3$.

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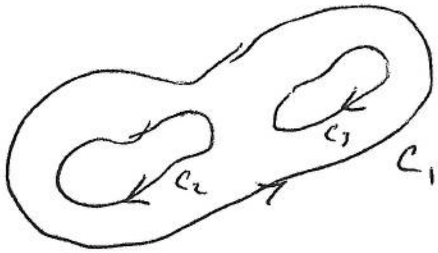

Orient each component so that D is always on the left.

Then Thm holds, with $\displaystyle\int_{\partial D} \vec F \cdot d\vec s = \int_{C_1} \vec F \cdot d\vec s + \int_{C_2} \vec F \cdot d\vec s + \int_{C_3} \vec F \cdot d\vec s$

⑥ In vector form, we note that for $\vec F = P(x,y)\vec i + Q(x,y)\vec j$,

the scalar curl of $\vec F$, $\cancel{\nabla \times \vec F = \frac{\partial P}{\partial y}}$

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$$\nabla \times \vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = 0\vec i - 0\vec j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec k$$

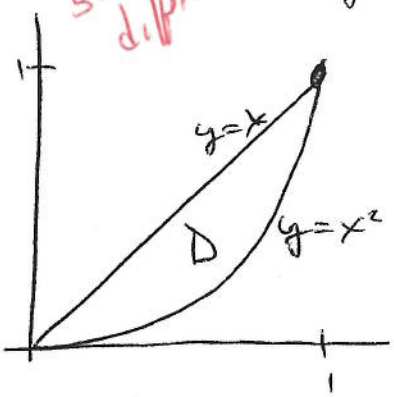
Hence thm says $\displaystyle\int_{\partial D} \vec F \cdot d\vec s = \iint_D (\nabla \times \vec F)\vec k \, dA$

since $(\nabla \times F) \cdot \vec k = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec k \cdot \vec k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

<u>Interpretation</u> How the vector field is helping a bead move along C is given by the aggregate twisting effect of $\vec F$ on D. $\nabla \times \vec F$ measures the twisting effect of $\vec F$ on D. We see this effect on the edge C.

ex 3 p. 435 but with different $\vec{F}$.

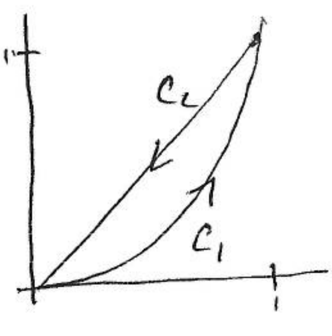
<u>ex</u> Let $\vec{F} = xy\vec{i} + y^2\vec{j}$ be defined on a region D in the first quadrant bounded by the functions $y = x$ and $y = x^2$.



Verify Green's Thm here.

<u>Strategy</u>: Parameterize the 2 segments of ∂D according to the orientation criterion for Green's Thm. Then calculate both integrals using the parameterization for ∂D and the fact that D is y simple: $D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 0 \le x \le 1 \\ x^2 \le y \le x \end{matrix} \right\}$.

<u>Solution</u>: Orient $\partial D = C$ counterclockwise and parameterize $C = C_1 + C_2$ compatible with orientation



$C_1: \begin{matrix} x = t \\ y = t^2 \end{matrix},\ 0 \le t \le 1 \qquad C_2: \begin{matrix} x = 1-t \\ y = 1-t \end{matrix} \quad 0 \le t \le 1$

Here $$\oint_C \vec{F} \cdot d\vec{s} = \int_{C_1} \left(\begin{bmatrix} xy \\ y^2 \end{bmatrix} \Bigg|_{\substack{x=t \\ y=t^2}} \cdot \begin{bmatrix} 1 \\ 2t \end{bmatrix} \right) dt$$
$$+ \int_{C_2} \left(\begin{bmatrix} xy \\ y^2 \end{bmatrix} \Bigg|_{\substack{x=1-t \\ y=1-t}} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) dt$$

ex. Solution (cont'd)

$$\oint \vec{F}\cdot d\vec{s} = \int_{C_1} \left(t(t^2) + t^4(2t)\right) dt + \int_{C_2} \left(-(1-t)^2 - (1-t)^2\right) dt$$

$$= \int_0^1 (t^3 + 2t^5)\, dt + \int_0^1 -2(1-t)^2\, dt$$

$$= \left(\frac{t^4}{4} + \frac{t^6}{3}\right)\Big|_0^1 + \int_0^1 (-2 + 4t - 2t^2)\, dt$$

$$= \frac{1}{4} + \frac{1}{3} + \left(-2t + 2t^2 - \frac{2}{3}t^3 \Big|_0^1\right)$$

$$= \frac{1}{4} + \frac{1}{3} - 2 + 2 - \frac{2}{3} = \frac{1}{4} - \frac{1}{3} = \frac{-1}{12}.$$

This should also equal by Green's Thm. $\vec{F} = \underbrace{xy}_{P(x,y)}\vec{i} + \underbrace{y^2}_{Q(x,y)}\vec{j}$

$$\iint_D \left(\frac{\partial(\overbrace{y^2}^{Q})}{\partial x} - \frac{\partial(\overbrace{xy}^{P})}{\partial y}\right) dx\,dy = \int_0^1 \int_{x^2}^{x} -x\, dy\, dx$$

$$= \int_0^1 \left(-xy\Big|_{y=x^2}^{y=x}\right) dx$$

$$= \int_0^1 -(x^2 - x^3)\, dx = -\left(\frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1\right) = \frac{-1}{12}.$$

A clever application

Thm The area of a bounded region D whose boundary is a simple closed curve C, is

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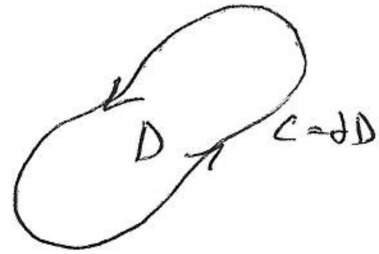
$$A(D) = \frac{1}{2}\oint_C x\,dy - y\,dx$$



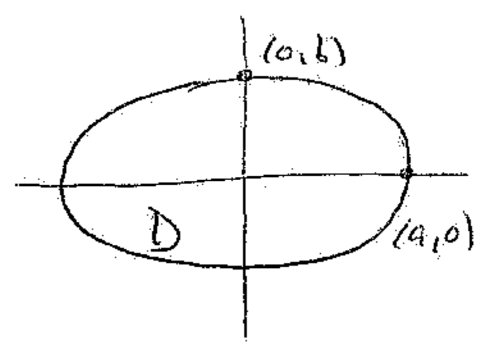
pf. Here $\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_{\partial D} \vec{F}\cdot d\vec{s}$, where $\vec{F} = -y\,\vec{i} + x\,\vec{j}$. By Green's Thm:

$$\frac{1}{2}\int_C \vec{F}\cdot d\vec{s} = \frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\iint_D \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dx\,dy$$

$$= \frac{1}{2}\iint_D 2\,dx\,dy = \iint_D dx\,dy$$

$$= \text{area}(D).$$

ex. Find the area of the ellipse at right.



Strategy: This ellipse is given by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Parameterize $C=\partial D$ and use the area version of Green's Thm:

Solution: Parameterize $C=\partial D$ counterclockwise via

$$x = a\cos t,\quad y = b\sin t, \quad \text{for } 0 \le t \le 2\pi.$$

Then $$\text{Area}(D) = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy = \frac{1}{2}\oint_{\partial D}\left(\begin{bmatrix}-y\\ x\end{bmatrix}\Bigg|_{\substack{x=a\cos t\\ y=b\sin t}} \cdot \begin{bmatrix}-a\sin t\\ b\cos t\end{bmatrix}\right)dt$$

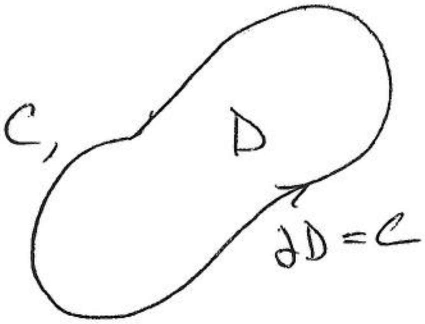
$$= \frac{1}{2}\int_0^{2\pi}(ab\sin^2 t + ab\cos^2 t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}\left(abt\Big|_0^{2\pi}\right) = \pi ab$$

Note: For an ellipse where $a=b$, we have a circle of radius a. Its area is πa^2.

Last note

For a domain D with boundary C,



the quantity $\int_C \vec{F} \cdot d\vec{s}$ is referred to as the circulation of $\vec{F}$ around C.

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It measures the tendency of $\vec{F}$ to move a bead along C in the oriented direction. If positive then the vector field is helping the bead to move, if negative hampering it.