Lecture 32.

And just like, for curves $\vec{c}:[a,b]\to\mathbb{R}^3$, we can define scalar and surface line integrals

$$\int_{\vec{c}} f\,ds \quad \text{and} \quad \int_{\vec{c}} \vec{F}\cdot d\vec{s}$$

via the parameterization $\vec{c}(t)$,

$$\Vert$$

$$\int_{\vec{c}} \cancel{f(\vec{x})}\, f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

and

$$\int_{\vec{c}} \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt$$

we can do the same for surfaces:

Def The <u>scalar surface integral</u> of $f:\mathbb{R}^3\to\mathbb{R}$ over a surface $S\subset\mathbb{R}^3$ is

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$$\iint_S f(\vec{x})\,dS = \iint_D f(\Phi(u,v))\,\|\vec{T}_u\times\vec{T}_v\|\,du\,dv$$

for $\Phi:D\to\mathbb{R}^3$, $\Phi(u,v)=(x(u,v),y(u,v),z(u,v))$, $S=\Phi(D)$.

<u>Notes</u> ① Remember that

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$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2}$$

So once the conversion to uv-coordinates is made, this is just a regular double integral over D, like before.

~~ex.~~ ② Integrating the function $f(x,y,z)=1$ over a surface gives just surface area.

ex. Integrate $f(x,y,z)=z$ over the standard torus from above.

This is just the scalar surface integral of f over $\mathbb{T}$:

$$\iint f\,dS = \iint_{\mathbb{T}} z\,dS = \iint_D f(x(u,v),y(u,v),z(u,v))\|\vec{T}_u\times\vec{T}_v\|\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} \cos v\underbrace{(2+\cos u)}_{\text{from surface area calculation.}}\,du\,dv$$

ex. (Cont'd)

$$\iint f\,dS = \int_0^{2\pi}\int_0^{2\pi} \cos v\,(2+\cos u)\,du\,dv$$

$$= \int_0^{2\pi} \cos v \int_0^{2\pi} (2+\cos u)\,du\,dv$$

$$= \int_0^{2\pi} \cos v \left(2u + \sin u\Big|_0^{2\pi}\right) dv$$

$$= \int_0^{2\pi} 4\pi \cos v\,dv = 4\pi\left(\sin v\Big|_0^{2\pi}\right) = 0$$

Big Question: Why do you think this quantity is 0??

A property of surfaces

. Orienting a curve is giving a sense of direction along the curve

This is different for a surface

as one can move in many directions along a surface. Instead, we define orientation via the normal vector to a surface:

Def. A surface orientation of S in $\mathbb{R}^3$, if it exists, is a choice of unit normal vector at each pt of the surface so that the vectors vary cont.

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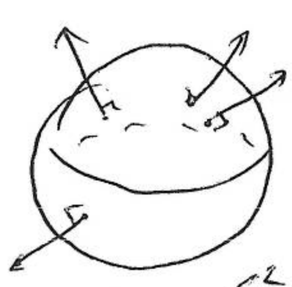
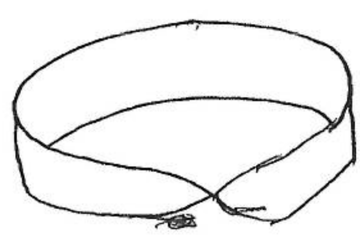
If a surface can have an orientation, we say it is orientable.



orientable surface



Möbius band is not orientable

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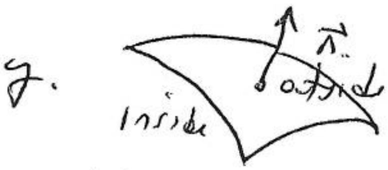
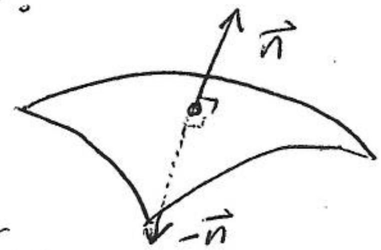
<u>Notes</u> ① An orientation is a declaration of inside/outside for a surface, or over vs. under. Like that for curves, there are exactly 2 orientations for a surface if there are any.



② At any pt of a surface, there are 2 unit normal vectors.



Any parameterization

$\Phi : D \to \mathbb{R}^3$ determines

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an orientation by computing $\vec{n}_\Phi = \dfrac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|}$

a unit normal vector at any regular point.

If $\Psi : D^* \to \mathbb{R}^3$ is another parameterization of

$$S = \Psi(D^*) = \Phi(D), \text{ w/ } \vec{n}_\Psi = \frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|}$$

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Ψ is orientation preserving if

$$\vec{n}_\Phi = \vec{n}_\Psi$$

and reversing if $\vec{n}_\Phi = -\vec{n}_\Psi$.

<u>Def</u> For $\vec{F}$ a vector field in $\mathbb{R}^3$, and ~~Φ~~ $S = \Phi(D)$, $\Phi : D \to \mathbb{R}^3$ a surface. The <u>vector surface integral</u> of $\vec{F}$ over S is

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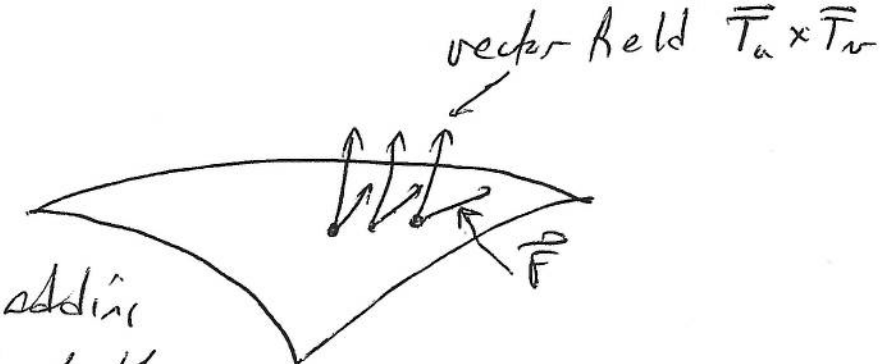
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v)\, du\, dv$$

It measures the flux of $\vec{F}$ across ~~the~~ S.

<u>Note</u>: (1) Here



we are adding up all of the dot products of F with $\vec{n} = T_u \times T_v$. If $\vec{F}$ and $\vec{n}$ are generally in the same direction then positive.

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The vector surface integral measures the aggregate flow of the flow lines of the ~~external~~ vector field from inside to outside.

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(2) Like scalar and vector line integrals, given $\Psi : D^* \to \mathbb{R}^3$ and $\Phi : D \to \mathbb{R}^3$ 2 parameterizations of $S = \Phi(D) = \Psi(D^*)$, where Ψ is orientation reversing wrt Φ,

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$$\iint_S f\,dS = \iint_{\Phi} f(\Phi(u,v))\,\|\vec{T}_u \times \vec{T}_v\|\,du\,dv$$

$$= \iint_{\Psi} f(\Psi(u,v))\,\|\vec{T}_u \times \vec{T}_v\|\,du\,dv$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\Phi} \vec{F}(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v)\,du\,dv$$

$$= -\iint_{\Psi} \vec{F}(\Psi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v)\,du\,dv.$$