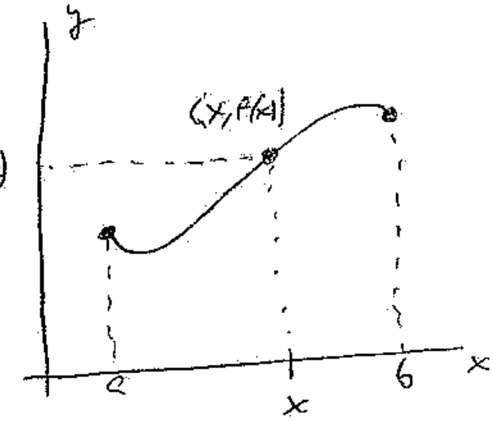
Lecture 30:

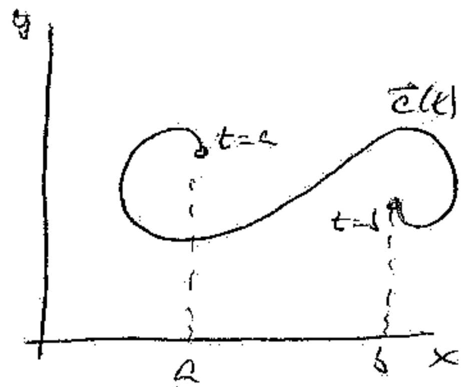
For any function $f:[a,b]\to\mathbb{R}$, we can use the variable itself to parameterize the $\text{graph}(f)\subset\mathbb{R}^2$:



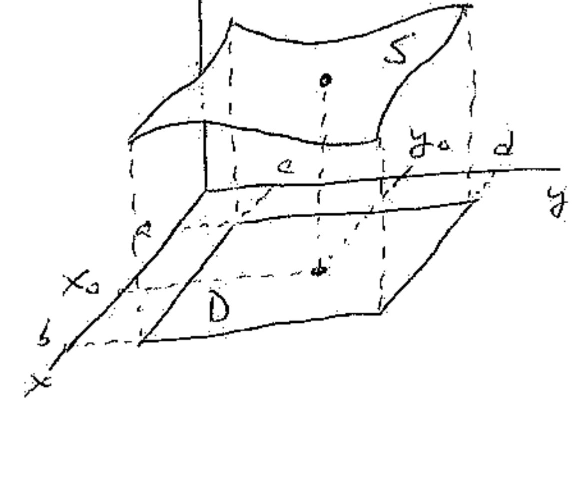
$\text{graph}(f) = \vec{c}:[a,b]\to\mathbb{R}^2$,

$\vec{c}(x)=\begin{bmatrix} x \\ f(x)\end{bmatrix}$. (or $\vec{c}(t)=\begin{bmatrix} t \\ f(t)\end{bmatrix}$).

But many curves in $\mathbb{R}^2$ cannot be written as graphs of functions. So we use $\vec{c}:[a,b]\to\mathbb{R}^2$ as a way to parameterize the curve.



The same is true for surfaces!



When $S=\text{graph}(f(x,y))$, the pts $(x,y)\in D$ can be used to uniquely determine pts on S via $(x,y,f(x,y))\in S\subset\mathbb{R}^3$.

However, many surfaces in $\mathbb{R}^3$ are not the graphs of functions in $\mathbb{R}^2$:

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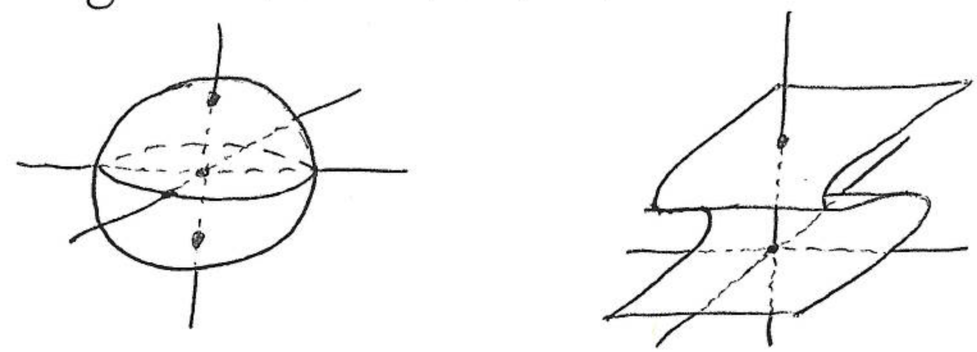

But we can still parameterize them (give them coordinates directly on the surfaces).

<u>Def</u> A <u>surface parameterization</u> (or the parameterization of a surface) is a function $\Phi: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^n$, $n \geq 3$ where D is a domain in $\mathbb{R}^2$. The surface itself is $S = \Phi(D)$, where

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$$\Phi(u,v) = (x_1(u,v), x_2(u,v), \dots, x_n(u,v)).$$

Note: Compare this to the definition of a curve $\vec{c}: [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^n$. The surface is simply a 2-dimensional version of $\vec{c}$.

ex. Let $\Phi : \overbrace{[0,2\pi]\times[0,2\pi]}^{D} \longrightarrow \mathbb{R}^3$,

$$\Phi(u,v) = \Big(\underbrace{(2+\cos u)\cos v}_{x(u,v)},\ \underbrace{(2+\cos u)\sin v}_{y(u,v)},\ \underbrace{\sin u}_{z(u,v)}\Big)$$

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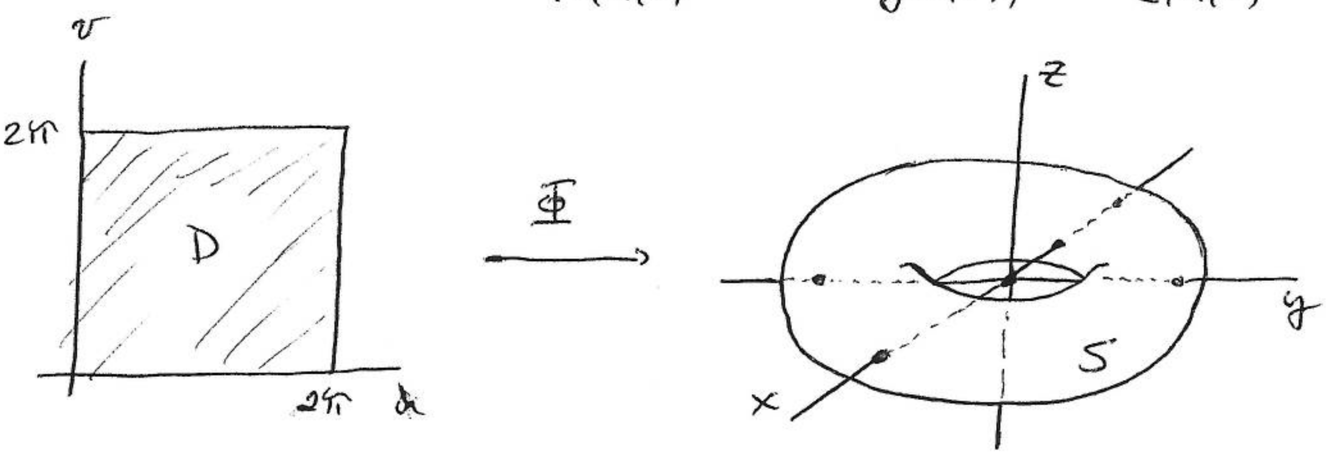
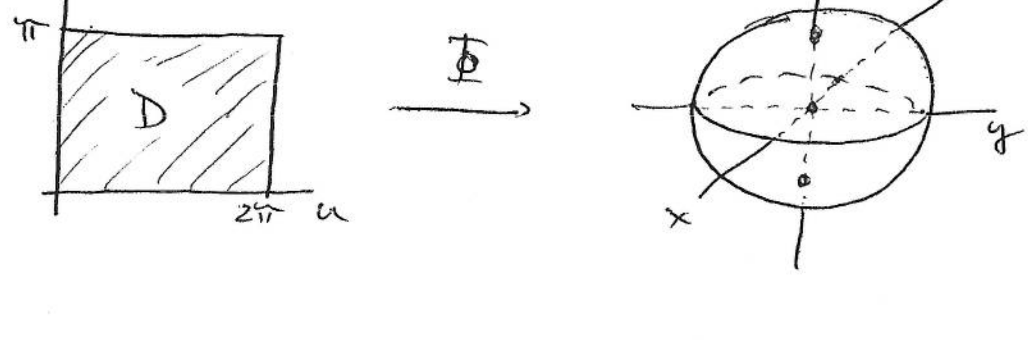


This surface is called a torus. This one has cross-sectional radius 1 and the center hole also is of radius 1.

ex. let $\Phi : [0,2\pi]\times[0,\pi] \longrightarrow \mathbb{R}^3$,

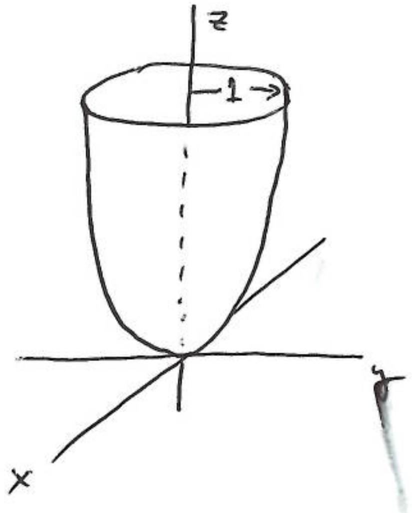
$$\Phi(u,v) = \big(\cos u \sin v,\ \sin u \sin v,\ \cos v\big)$$

(these are spherical coordinates



Note: Like curves, surfaces can be defined by maps that are not one-to-one, but only in certain ways.

ex.



Here, $\Phi : D \to \mathbb{R}^3$, $D =$ unit disk in $\mathbb{R}^2$

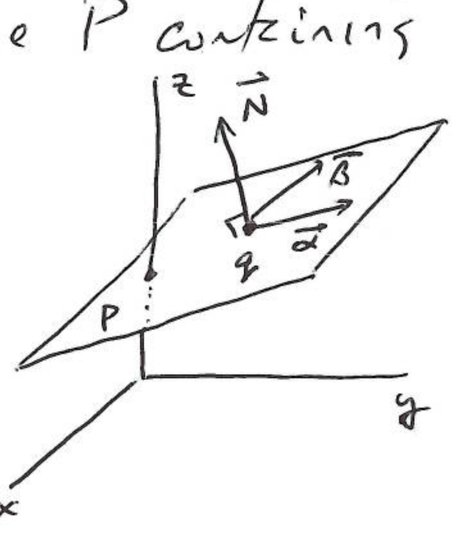
and $\Phi(u,v) = (u, v, u^2+v^2) \in \mathbb{R}^3$

ex. Choose a pt $q = (x_0, y_0, z_0) \in \mathbb{R}^3$ and any 2 vectors, nontrivial and non-collinear, based at q, $\vec{\alpha}$ and $\vec{\beta}$. Then $\vec{\alpha} \times \vec{\beta} = \vec{N} = A\vec{i} + B\vec{j} + C\vec{k}$ is a vector, based at q, normal to both $\vec{\alpha}$ and $\vec{\beta}$, and

ex 1 p. 377

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

is the equation of the plane P containing both $\vec{\alpha}$ and $\vec{\beta}$.

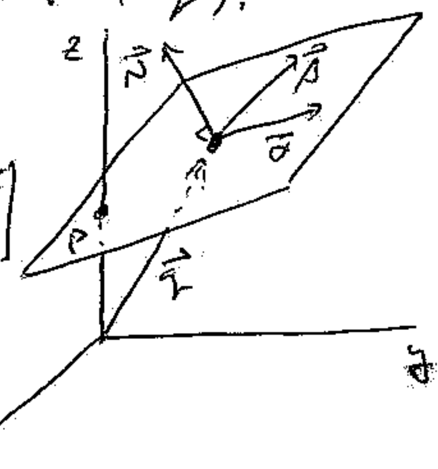


Interpretations of the Plane P:

$$P = \{ \text{vectors } \vec{V} \in \mathbb{R}^3, \text{ based at } q, \text{ where } \vec{V} \cdot \vec{N} = 0 \}$$

$$= \{ \text{vectors } \vec{V} \in \mathbb{R}^3, \text{ based at } q, \text{ where } \vec{V} = a\vec{\alpha} + b\vec{\beta},\ a, b \in \mathbb{R} \}.$$

Consider now $\vec{q} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ the vector version of the pt. q (based at the origin with head at q).



Then, any point $(x,y,z) \in \mathbb{R}^3$ will be in P if the vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ based at the origin has components

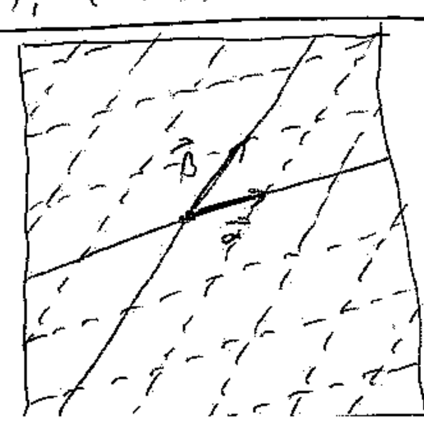
$$\vec{x} = \vec{q} + a\vec{\alpha} + b\vec{\beta}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + a \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + b \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \text{ for } (a,b) \in \mathbb{R}^2.$$

Thus, we can parameterize P as a surface: Let

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^3, \quad \Phi(u,v) = \vec{q} + u\vec{\alpha} + v\vec{\beta}$$
$$= (x_0 + u\alpha_1 + v\beta_1,\ y_0 + u\alpha_2 + v\beta_2,\ z_0 + u\alpha_3 + v\beta_3)$$
$$= (x(u,v),\ y(u,v),\ z(u,v))$$

Note: Parameterizing P via $\vec{\alpha}$ and $\vec{\beta}$ puts a coord. system on P. It looks like $\vec{\alpha}, \vec{\beta}$ are unit vectors of a skewed rectilinear grid on P.



So suppose we have a surface in $\mathbb{R}^3$:

$\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$

If Φ is C^1, then $S = \Phi(D)$ has tangent planes at its points in $\mathbb{R}^3$.

Q: Can we write the tangent plane to S in terms of its parameterization?

Note: We did this for a C^1-curve $\vec{c}: [a,b] \to \mathbb{R}^n$ using $\vec{c}\,'(t)$: Here $l: \mathbb{R} \to \mathbb{R}^n$, $l(t) = c(t_0) + c'(t_0)(t - t_0)$ was the equation of the tangent line to $\vec{c}(t)$ at $t = t_0$.

A: Yes. Here $D\Phi$ is a 3×2 matrix $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$.

Choose a pt $(u_0, v_0) \in D$. If we fix the v coordinate $v = v_0$, we can form a curve in D along the fixed $v = v_0$ line.

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Its image under Φ is a curve in the surface. The tangent vector to this curve at (u_0, v_0) is

$$\vec{T}_{u_0} = \frac{\partial \Phi}{\partial u}(u_0,v_0) = \frac{\partial x}{\partial u}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\vec{k}$$

and in the v-direction (fixing $u=u_0$),

$$\vec{T}_{v_0} = \frac{\partial \Phi}{\partial v}(u_0,v_0) = \frac{\partial x}{\partial v}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\vec{k}$$

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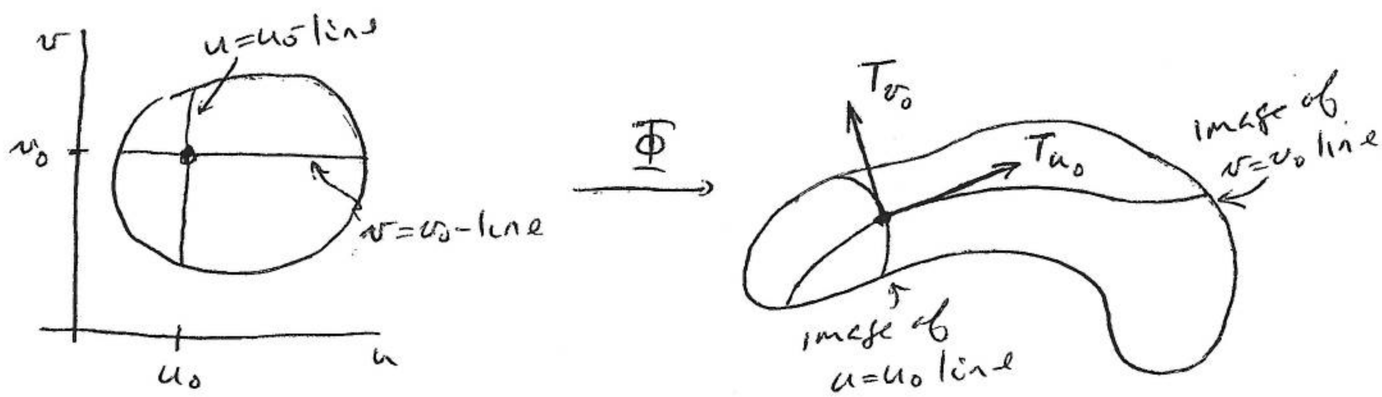

These 2 vectors span a plane when they are both non-zero, and not dependent. This is the tangent plane to $\Phi(D)$ at $\Phi(u_0,v_0)\in S$.

This plane can be written using the components of the normal vector to these, $\vec{N} = \vec{T}_{u_0}\times\vec{T}_{v_0}$
$= N_1\vec{i} + N_2\vec{j} + N_3\vec{k}$

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Def If a parameterized surface $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is regular at $\Phi(u_0, v_0)$ (basically, this means $\vec{T}_{u_0} \times \vec{T}_{v_0} \neq \vec{0}$ at (u_0, v_0)), then for $\vec{N} = \vec{T}_{u_0} \times \vec{T}_{v_0} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$, the equation of the plane tangent to the surface at $\Phi(u_0, v_0) = (x_0, y_0, z_0)$ is

$$\vec{N} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = N_1(x - x_0) + N_2(y - y_0) + N_3(z - z_0) = 0$$

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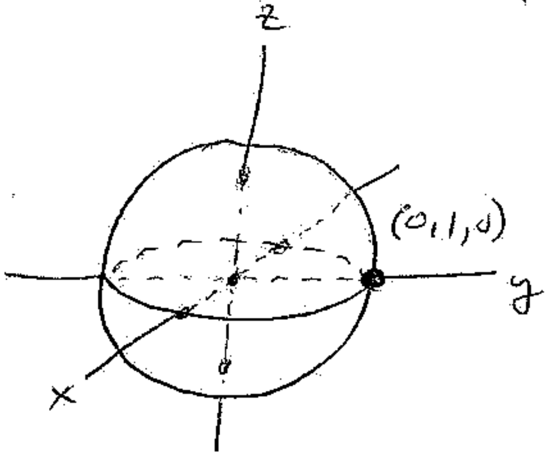
ex. Compute the tangent space to the unit sphere at the pt $(0, 1, 0) \in \mathbb{R}^3$.

Strategy: We parameterize the unit sphere using the above spherical coordinates, solve for (u_0, v_0), where $\Phi(u_0, v_0) = (0, 1, 0)$ and calculate the tangent space using the above definition.

Solution Here, $\Phi : D \to \mathbb{R}^3$, $D = [0, 2\pi] \times [0, \pi]$

$$\Phi(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$$

$$= (0, 1, 0) \text{ when } u = u_0 = \frac{\pi}{2} = v_0 = v.$$



Here $\vec{T}_{u_0} = \vec{T}_{\frac{\pi}{2}} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \Big|_{\substack{u = \frac{\pi}{2} \\ v = \frac{\pi}{2}}}$

$$= (-\sin u \sin v, \cos u \sin v, 0)$$

$$= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

and $\vec{T}_{v_0} = \vec{T}_{\frac{\pi}{2}} = (\cos u \cos v, \sin u \cos v, -\sin v)$

$$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

And $\vec{N} = \vec{T}_{u_0} \times \vec{T}_{v_0} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -\vec{j} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$

Hence the equation of the tangent plane to the unit sphere at $(0, 1, 0)$ is

$$\vec{N} \cdot \begin{bmatrix} x \\ y - 1 \\ z \end{bmatrix} = -1(y - 1) = \boxed{1 - y = 0}$$

(All pts (x, y, z) where the y-coordinate is 1). ∎

Last note: Given a parameterized surface, when $\vec{T}_{u_0} \times \vec{T}_{v_0} \neq \vec{0}$ at (u_0, v_0), then you know the tangent plane exists. Thus you know the surface is smooth there. It can be a nice test for smoothness.

ex 4 p380 Given $g: \mathbb{R}^2 \to \mathbb{R}$, write $S = \text{graph}(g)$

using (x,y) as its parameters on S:

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$$\Phi(u,v) \longrightarrow \mathbb{R}^3, \quad \Phi(u,v) = (\underbrace{u}_{x}, \underbrace{v}_{y}, \underbrace{g(u,v)}_{z}).$$

Then $\vec{T}_{u_0} = \frac{\partial x}{\partial u}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)$

$= \vec{i} + \frac{\partial g}{\partial u}(u_0,v_0)\vec{k}$

Similarly $\vec{T}_{v_0} = \vec{j} + \frac{\partial g}{\partial v}(u_0,v_0)\vec{k}$

So that $\vec{N} = \vec{T}_{u_0} \times \vec{T}_{v_0} = -\frac{\partial g}{\partial u}(u_0,v_0)\vec{i} - \frac{\partial g}{\partial v}(u_0,v_0)\vec{j} + k.$

Here $\vec{N}$ cannot be the $\vec{0}$ vector, since the last component is always nonzero.

Hence when $g(x,y)$ is differentiable as a function, its graph is always smooth.