Lecture 29.

A few more interesting facts about line integrals and curves:

(I) The Fundamental Thm of Calculus (Calc I) implies

Thm 3 p. 367

<u>Thm</u> for $f:\mathbb{R}^n \to \mathbb{R}$ a C^1-function, and $\vec{c}:[a,b] \to \mathbb{R}^n$ a piecewise C^1-curve

$$\int \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

example 9 pg 367 is a great example.

Here is another:

ex. Evaluate $\int_{\vec{c}} yz\,dx + xz\,dy + xy\,dz$ over

$\vec{c}(t) = \frac{t^4}{4}\vec{i} + \sin\left(\frac{t\pi}{2}\right)\vec{j} + t\vec{k}$ on $[0,1]$.

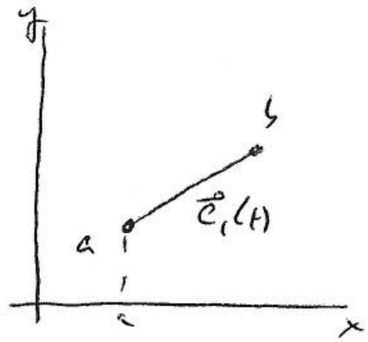
Note: Often written this way, $yz\,dx + xz\,dy + xy\,dz$ is just $\vec{F}\cdot d\vec{s}$, where $\vec{F} = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$, $d\vec{s} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$.

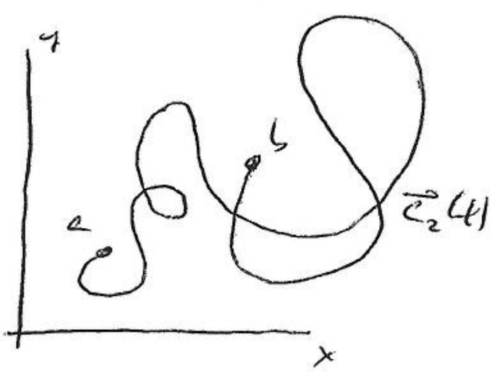
Here $\vec{F} = \nabla f$, where $f(x,y,z) = xyz$. Thus by the thm

$$\int \vec{F}\cdot d\vec{s} = f(\vec{c}(1)) - f(\vec{c}(0)) = f(\tfrac{1}{4},1,1) - f(0,0,0) = \tfrac{1}{4}.$$

So integrating a gradient vector field over a curve only depends on the endpts of the curve!

For $\vec{F} = \nabla f$, $f : \mathbb{R}^2 \to \mathbb{R}$,





Here $\displaystyle\int_{\vec{c}_1} \vec{F} \cdot d\vec{s} = \int_{\vec{c}_2} \vec{F} \cdot d\vec{s} = f(c(b)) - f(c(a)).$

(II) Defs

- A curve $\vec{c} : [a,b] \to \mathbb{R}^n$ is called <u>simple</u> if the map is 1-1 (image does not intersect itself).
- A C^0-curve $\vec{c} : [a,b] \to \mathbb{R}^n$ is called closed if $\vec{c}(b) = \vec{c}(a)$.
- A C^0-curve is simple closed if $\vec{c}$ is 1-1 on $[a,b)$ and $c(b) = c(a)$

p. 368

- A sense of direction on a simple curve is called an <u>orientation</u>

Note: There are 2 possible orientations for a simple curve. Any parameterization induces an orientation.

(III) For quantities where the parameterization does not matter, we sometimes only speak of the curve:

ex: Calculate $\int_C f\,ds$ where $f(x,y)=\sqrt{1+x^2}$

over $C=\{(x,y) \mid y=3x^2,\ x\in[0,2]\}$

For quantities where direction does matter, we still write the integral over C, but include either a parameterization, or at least an orientation:

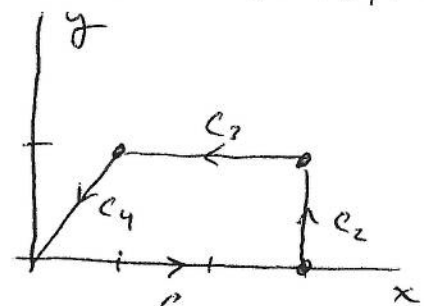
ex. Calculate $\int_C \vec{F}\cdot d\vec{s}$ for C as above going from $(0,0)$ to $(2,12)$. OR parametrize C via $\vec{c}:[0,2]\to\mathbb{R}^2$, $\vec{c}=\begin{bmatrix} t \\ 3t^2 \end{bmatrix}$. Then $\int_C \vec{F}\cdot d\vec{s}=\int_{\vec{c}} \vec{F}\cdot d\vec{s}$.

(IV) Suppose an oriented C consists of several pieces C_i, $i=1,\dots,k$.

Then $$\int_C \vec{F}\cdot d\vec{s} = \int_{C_1} \vec{F}\cdot d\vec{s} + \cdots + \int_{C_k} \vec{F}\cdot d\vec{s} = \sum_{i=1}^{k} \int_{C_i} \vec{F}\cdot d\vec{s}$$

This way, one can parameterize each piece separately!

ex. Calculate $\int_C (x^2-y)\,dx + (x-y^2)\,dy$, where C is the trapezoid with vertices at $(0,0)$, $(3,0)$, $(3,1)$, $(1,1)$, oriented counterclockwise.



Similar to ex. 11, p. 371

Strategy: Parameterize the 4 paths at right and evaluate $\int_{C_i} \vec{F}\cdot d\vec{s}$, $\vec{F} = \begin{bmatrix} x^2-y \\ x-y^2 \end{bmatrix}$.

Solution: Parameterize the 4 paths:

C_1: $(t,0)$, $0\le t\le 3$, so $\vec{c}_1(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$, $\vec{c}_1{}'(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

C_2: $(3,t)$, $0\le t\le 1$, so $\vec{c}_2(t) = \begin{bmatrix} 3 \\ t \end{bmatrix}$, $\vec{c}_2{}'(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

C_3: $(3-t,1)$, $0\le t\le 2$, so $\vec{c}_3(t) = \begin{bmatrix} 3-t \\ 1 \end{bmatrix}$, $\vec{c}_3{}'(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

C_4: $(1-t,1-t)$, $0\le t\le 1$, so $\vec{c}_4(t) = \begin{bmatrix} 1-t \\ 1-t \end{bmatrix}$, $\vec{c}_4{}'(t) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

Then $$\int_{C_1} \vec{F}\cdot d\vec{s} = \int_{\vec{c}_1} \left(\begin{bmatrix} x^2-y \\ x-y^2 \end{bmatrix}\Bigg|_{\substack{x=t\\ y=0}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) dt$$

$$= \int_0^3 t^2\, dt = \frac{t^3}{3}\Big|_0^3 = 9.$$

$$\int_{C_2} \vec{F}\cdot d\vec{s} = \int_{\vec{c}_2} \left(\begin{bmatrix} x^2-y \\ x-y^2 \end{bmatrix}\Bigg|_{\substack{x=3\\ y=t}} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) dt$$

$$= \int_0^1 (3-t^2)\, dt = 3t - \frac{t^3}{3}\Big|_0^1 = 3 - \frac{1}{3} = \frac{5}{3}.$$

$$\int_{C_3} \vec{F}\cdot d\vec{s} = \int_{\vec{c}_3} \left(\begin{bmatrix} x^2-y \\ x-y^2 \end{bmatrix}\Bigg|_{\substack{x=3-t\\ y=1}} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) dt$$

$$= \int_0^2 -((3-t)^2-1)\, dt = -\int_0^2 (8-6t+t^2)\, dt$$

$$= \left(-8t + 3t^2 - \frac{t^3}{3}\right)\Big|_0^2 = -16+12-\frac{8}{3} = \frac{-20}{3}.$$

$$\int_{C_4} \vec{F}\cdot d\vec{s} = \int_{\vec{c}_4} \left(\begin{bmatrix} x^2-y \\ x-y^2 \end{bmatrix}\Bigg|_{\substack{x=1-t\\ y=1-t}} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) dt$$

$$= \int_0^1 \left(-(1-t)^2 + (1-t) - (1-t) + (1-t)^2\right) dt = 0$$

Hence $$\int_C \vec{F}\cdot d\vec{s} = 9 + \frac{5}{3} - \frac{20}{3} + 0 = 9 - 5 = 4 \qquad \blacksquare$$