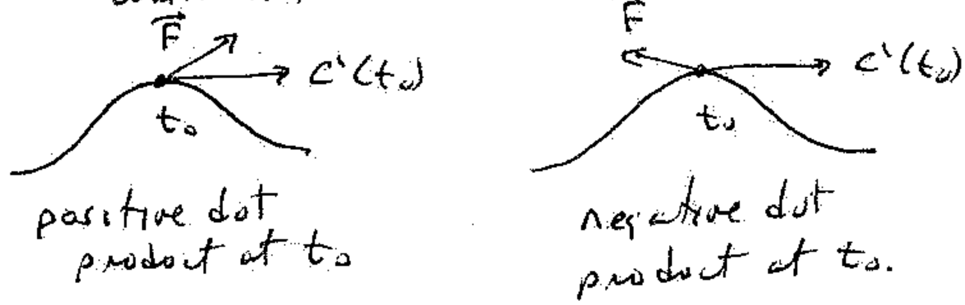
Lecture 28

Notes ① At each pt along $\vec{c}$, the integrand is the dot product of the vector field vector and the velocity vector of $\vec{c}$.

It is positive when vectors are more or less in the same direction, and negative when not:



In some sense, it measures how much a bead on the wire must fight against a vector field while traversing the wire.

It measures the work done by the vector field (force field) on the bead.

② For $\vec{F}(\vec{x}) = \begin{bmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{bmatrix}$, and $d\vec{s} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$, it is

common to see

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} F_1\,dx_1 + \cdots + F_n\,dx_n$$

ex. $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F}(\vec{x}) = x^2\vec{i} + xy\vec{j} + \vec{k}$,

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} x^2\,dx + xy\,dy + dz$$

ex: Let $\vec{F} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$ be a vector field on $\mathbb{R}^3$, and $\vec{c}(t) = \begin{bmatrix} t \\ 3t^2 \\ 2t^3 \end{bmatrix}$ be defined on $[0,1]$. ~~Ca~~ Calculate the vector line integral of $\vec{F}$ on $\vec{c}$.

Solution: Here $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$

$$= \int_0^1 \left(\begin{bmatrix} 2t \\ 6t^2 \\ 4t^3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 6t \\ 6t^2 \end{bmatrix} \right) dt$$

$$= \int_0^1 (2t + 36t^3 + 24t^5)\, dt$$

$$= \left(t^2 + 9t^4 + 4t^6 \Big|_0^1 \right) = 14$$

Another Interpretation

For a C^1-path $\vec{c}: [a,b] \longrightarrow \mathbb{R}^n$, define its unit tangent vector at each pt in $\vec{c}$:

$$\vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}$$

Another Interpretation (cont'd)

Then, for a vector field $\vec{F}$, the vector line integral along $\vec{c}$ is

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} \|\vec{c}\,'(t)\|\,dt$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{T}(t)\,\|\vec{c}\,'(t)\|\,dt$$

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{T}(t)\,ds = \int_{\vec{c}} (\vec{F} \cdot \vec{T})\,ds$$

The vector line integral of a vector field $\vec{F}$ along a curve $\vec{c}$ is equal to the scaler line integral of the component of $\vec{F}$ in the tangential direction of $\vec{c}$, $(\vec{F} \cdot \vec{T})$.

This will be useful later on.

VII

<u>Def</u> Let $h: I \to J$ be a C^1-real valued function which is one-to-one on $I = [a,b] \subset \mathbb{R}$, and onto $J = [c,d]$. For $\vec{c}: J \to \mathbb{R}^n$ a piecewise C^1-curve, the composition

$$\vec{p} = \vec{c} \circ h : I \to \mathbb{R}^n$$

is called a reparameterization of $\vec{c}$.

<u>Note:</u> There are 2 types of reparameterizations

- orientation preserving: $h(a)=c,\ h(b)=d$ (same direction)
- orientation reversing: $h(a)=d,\ h(c)=c$ (opp. direction)

Q: What effect does a reparameterization have on a vector line integral? A scalar line integral?

A: Not much!

<u>Thm</u> Let $\vec{F}$ be a C^0-vector field and f a C^0-function on a domain that contains a piecewise C^1-curve $\vec{c}: [a,b] \to \mathbb{R}^n$. Let $\vec{p}: [c,d] \to \mathbb{R}^n$ be <u>any</u> reparameterization. Then

(A) $\int_{\vec{c}} f\, ds = \int_{\vec{p}} f\, ds$

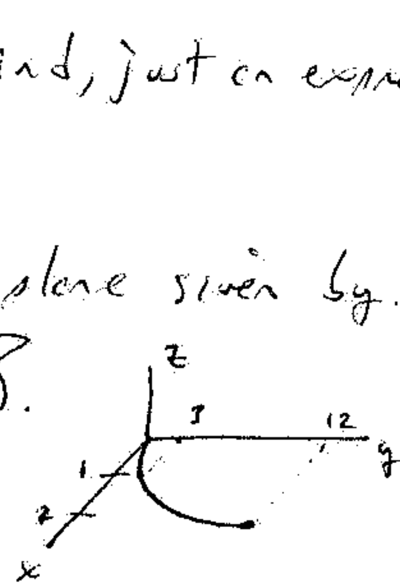
(B) $\int_{\vec{p}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} \vec{F} \cdot d\vec{s}$ if $\vec{p}$ orientation preserving

$\int_{\vec{p}} \vec{F} \cdot d\vec{s} = -\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ if $\vec{p}$ orientation reversing

Final Notes: Often a curve is written without a particular parameterization in mind, just an expression for the curve.

ex. Let C be a curve in the plane given by

$$C = \{(x,y) \in \mathbb{R}^2 \mid y = 3x^2\}.$$



Then an obvious parameterization for C is

$$\vec{c} : [0,2] \longrightarrow \mathbb{R}^2, \quad \vec{c}(t) = \begin{bmatrix} t \\ 3t^2 \end{bmatrix}.$$

An orientation reversing reparameterization for $\vec{c}$ is

$$\vec{p} : [0,1] \longrightarrow \mathbb{R}^2, \quad \vec{p}(t) = \begin{bmatrix} 2\sqrt{1-t} \\ 12(1-t) \end{bmatrix}$$

Say we wanted to calculate the work done by the vector field $\vec{F} = x\vec{i} - y\vec{j} + (x+y+z)\vec{k}$ on a bead traversing C from $(0,0,0)$ to $(2,12,0)$.

Any parameterization of C as a curve in $\mathbb{R}^3$ (with z-coordinate 0) will do, although we must account for the orientation of the curve C given by the parameterization.

Check this: We need to go from $(0,0,0)$ to $(2,12,0)$ along C.

First, choose $\vec{c}:[0,2] = \begin{bmatrix} t \\ 3t^2 \\ 0 \end{bmatrix}$ as our parameterization.

Then $\text{Work} = \int_C \vec{F}\cdot d\vec{s} = \int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_0^2 \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt$

$$= \int_0^2 \left(\begin{bmatrix} t \\ -3t^2 \\ t+3t^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 6t \\ 0 \end{bmatrix} \right) dt$$

$$= \int_0^2 (t - 18t^3)\,dt = \left(\frac{t^2}{2} - \frac{9}{2}t^4 \Big|_0^2 \right)$$

$$= 2 - 72 = -70. \blacksquare$$

Next, choose $\vec{p}:[0,1] = \begin{bmatrix} 2\sqrt{1-t} \\ 12(1-t) \\ 0 \end{bmatrix}$, knowing that we are going the wrong way along C:

Then $\text{Work} = \int_C \vec{F}\cdot d\vec{s} = -\int_{\vec{p}} \vec{F}\cdot d\vec{s} = -\int_0^1 \vec{F}(\vec{p}(t))\cdot \vec{p}\,'(t)\,dt$

$$= -\int_0^1 \left(\begin{bmatrix} 2\sqrt{1-t} \\ -12(1-t) \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{\sqrt{1-t}} \\ -12 \\ 0 \end{bmatrix} \right) dt$$

$$= -\int_0^1 (-2 + 144(1-t))\,dt = \int_0^1 (144t - 142)\,dt$$

$$= \left(72t^2 - 142t \Big|_0^1 \right) = 72 - 142 = -70 \blacksquare$$