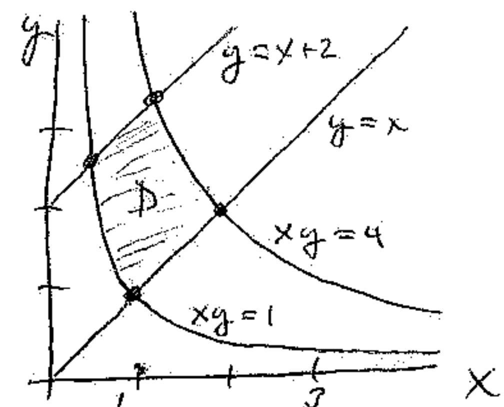
Lecture 27



Note: ② Sometimes, it is advantageous to, instead of writing $x, y$ as functions of 2 new variables $u, v$, to instead write new variables as functions of $x, y$ to simplify a domain. This is backwards to the theorem though.

ex. Integrate $\iint_D (x^2 - y^2) e^{xy} \, dx \, dy$, where $D$ is given here:



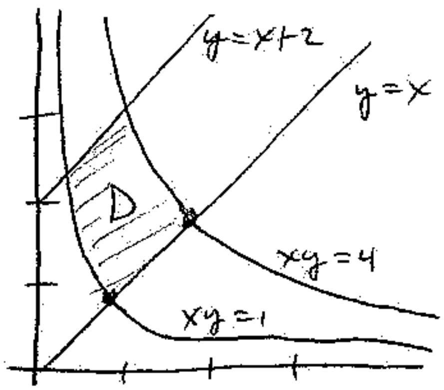

Strategy: We notice here that $D$ is bounded by 2 function types:

$$xy = \text{const. and } y - x = \text{const.}$$

Thus we substitute $u = xy$, $v = y - x$ and integrate.

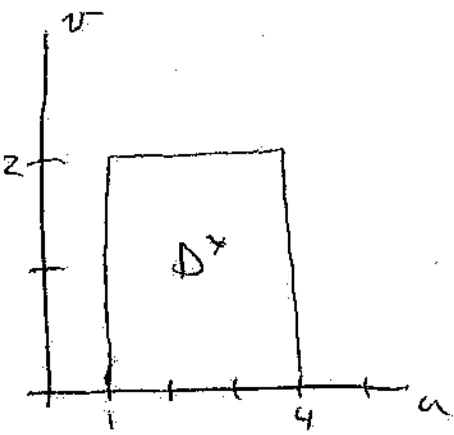
ex. Solution

Let $u = xy$, $v = y - x$. Then across $D$, $u$ ranges from $u=1$ to $u=4$ and $v$ ranges from $v=0$ to $v=2$. Hence




$$T: D \to D^*$$
$$\underrightarrow{T(x,y) = (xy,\, y-x)}$$

Here $u = xy$, $v = y - x$




But here, $T$ goes the wrong way to use the theorem:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

So instead of trying to find $T^{-1}: D^* \to D$ (solve for $x, y$ in terms of $u, v$) we do 2 things:

(a) Write $f$ in terms of $u, v$, and

(b) Calculate $\dfrac{\partial(x,y)}{\partial(u,v)} = \left[ \dfrac{\partial(u,v)}{\partial(x,y)} \right]^{-1}$ since $D(T^{-1}) = (DT)^{-1}$ by the Chain Rule.

First, we do (b):

Calculate $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -1 & 1 \end{vmatrix} = x+y$

Then $\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{x+y}$.

Then, for (a)

$$\iint_D (x^2-y^2)e^{xy}\,dx\,dy = \iint_{D^*} \cancel{(x+y)}(x-y)\,e^{xy}\left(\frac{1}{\cancel{x+y}}\right) du\,dv$$

$$= \iint_{D^*} -v e^u \,du\,dv$$

$$= \int_0^2 \int_1^4 -v e^u \,du\,dv$$

$$= \int_0^2 \left(-v e^u\Big|_1^4\right) dv = \int_0^2 -v(e^4-e)\,dv$$

$$= -\frac{v^2}{2}(e^4-e)\Big|_0^2 = 2(e-e^4). \quad \blacksquare$$

Alternatively, one could solve for $x, y$ as functions of $u, v$, thereby computing $T^{-1}: D^* \to D$ directly. But this is hard.

The change of Variables formula for triple integrals is similar: Given $\iiint_R f(x,y,z)\,dV$ to evaluate, if there is a change of variables $T: R^* \to R$, where $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

where $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is the Jacobian Determinant of $T$,

p. 323

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Notice the special cases for spherical and cylindrical coordinates.

At this point we begin to use the machinery we have developed to say and calculate some important results:

To start, let $\vec{c}:[a,b]\to\mathbb{R}^n$ be a $C^1$-curve (at worst, $\vec{c}$ needs only to be piecewise $C^1$). If $\mathbb{R}^n$ has a $C^0$-function $f:\mathbb{R}^n\to\mathbb{R}$ on it, or if $\mathbb{R}^n$ has a vector field $\vec{F}:\mathbb{R}^n\to\mathbb{R}^n$, we can "sum" the values of $f$ over $\vec{c}$, or measure the effect of $\vec{F}$ on $\vec{c}$ via integration:

<u>Def 1</u> The <u>scalar</u> <u>line</u> <u>integral</u> (book uses path integral) of $f:\mathbb{R}^n\to\mathbb{R}$ along $\vec{c}:[a,b]\to\mathbb{R}^n$ is

p. 352.

$$\int_{\vec{c}} f\,ds = \int_{\vec{c}} f(\vec{x})\,ds = \int_a^b f(\vec{c}(t))\,\overbrace{\|\vec{c}\,'(t)\|\,dt}^{ds\ (\text{see pg } 231)}$$

$$= \int_a^b f(x_1(t),\cdots,x_n(t))\,\|\vec{c}\,'(t)\|\,dt$$

Notes: ① When $f(\vec{x}) = 1$, then this integral is simply the arclength of $\vec{c}$ from $t=a$ to $t=b$:

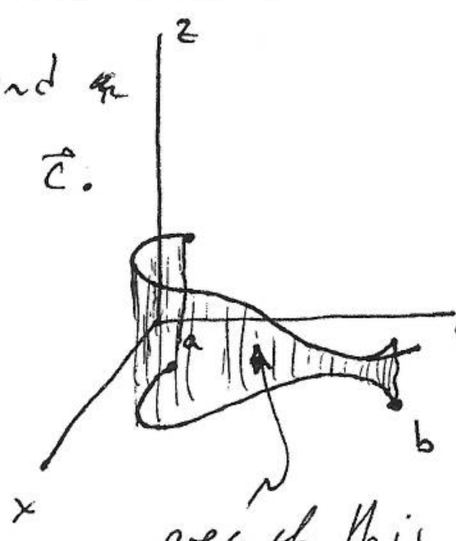
$$\int_{\vec{c}} 1\,ds = \int_{\vec{c}} ds = \int_a^b \|\vec{c}\,'(t)\|\,dt$$

② It is only necessary for $f$ to be defined along $\vec{c}$ (and not on all of $\mathbb{R}^n$) for this to work.

③ When $n=2$ (a planar curve), there is a nice geometric interpretation:

Let $\vec{c}:[a,b] \to \mathbb{R}^2$, $\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, and $f:\mathbb{R}^2 \to \mathbb{R}$, the graph$(f) \subset \mathbb{R}^3$. Place $\vec{c}(t)$ in the xy-plane of $\mathbb{R}^3$, and graph $f$ only along $\vec{c}$.

Then $\int_{\vec{c}} f\,ds$ is the area of the "fence" with height along $\vec{c}$.




area of this "fence" is $\int_{\vec{c}} f\,ds$

p. 354.

Think of this as a curved Calculus I integral of $f$ over $[a,b]$.

ex Let $\vec{c}:[0,2\pi]$ be the helix $\vec{c}(t)=\begin{bmatrix}\cos t\\ \sin t\\ t\end{bmatrix}$, and $f(x)=xy+z$. Then

$$\int_{\vec{c}} f\,ds=\int_0^{2\pi} f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

ex. 1 p. 352 with a different fnc.

We do this in pieces:

- $\|\vec{c}\,'(t)\|=\sqrt{(-\sin t)^2+(\cos t)^2+(1)^2}=\sqrt{2}$
- $f(\vec{c}(t))=(\cos t)(\sin t)+t$

Then
$$\int_{\vec{c}} f\,ds=\int_0^{2\pi}(\cos t\sin t+t)\sqrt{2}\,dt$$
$$=\int_0^{2\pi}\sqrt{2}\sin t\cos t\,dt+\int_0^{2\pi}\sqrt{2}\,t\,dt$$
$$=\sqrt{2}\left(\frac{\sin^2 t}{2}\Big|_0^{2\pi}\right)+\sqrt{2}\left(\frac{t^2}{2}\Big|_0^{2\pi}\right)=2\sqrt{2}\,\pi^2.$$

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<u>Def 2</u> Let $\vec{F}$ be a $C^0$-vector field defined along a $C^1$ curve $\vec{c}:[a,b]\to\mathbb{R}^n$. Then the <u>vector line integral</u> (book uses line integral) of $\vec{F}$ along $\vec{c}$ is

p. 359.

$$\int_{\vec{c}}\vec{F}\cdot d\vec{s}=\int_a^b \vec{F}(\vec{c}(t))\overset{\text{dot product}}{\cdot}\overbrace{\vec{c}\,'(t)\,dt}^{d\vec{s}\ (\text{see pg } 231)}$$