Lecture 26

X

ex. For $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

here $x = 2u + v$, $y = u + v$.

Then $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$$

Note: $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = \det(A)$, for $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

We have the following:

<u>Thm</u> (Change of Variables formula).

Let D^* and D be elementary regions in $\mathbb{R}^2$ and let $T: D^* \to D$ be a C^1 transformation which is one-to-one and onto at least the interior of D^* and D. Then for $f: D \to \mathbb{R}$,

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$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

<u>Note</u>: ① This is precisely the 2-dimensional Substitution Method: Let $x = x(u,v)$
$y = y(u,v)$.

Then $dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

② These are absolute value signs. They are necessary and related to the transformation.

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③ For the transformation from polar to rectilinear: $x = r\cos\theta$, $y = r\sin\theta$, note that $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$.

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Thus $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

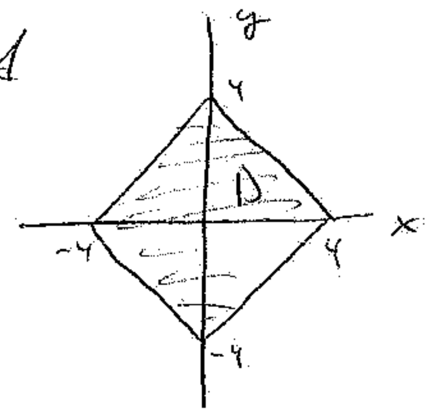
Do you recognize this? Calculate the area of a circle of radius r_0:

$$\int_0^{2\pi}\int_0^{r_0} 1\cdot r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^2}{2}\Big|_0^{r_0}\right)d\theta = \int_0^{2\pi}\frac{1}{2}r_0^2\,d\theta = \frac{1}{2}r_0^2\theta\Big|_0^{2\pi} = \pi r_0^2.$$

Let $f(x,y) = x^2$, defined on the domain

and suppose you wanted

the quantity $\iint_D f(x,y)\,dA$.



We will do this 2 ways:

(I) The edges of the domain D are given by the four equations $x+y=4$, $x-y=4$

$x+y=-4$, $x-y=-4$

(You should verify these). Write D as y-simple, so $D = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} -4 \le x \le 4 \\ \varphi_1(x) \le y \le \varphi_2(x) \end{array} \right\}$

where $\varphi_1(x) = \begin{cases} -x-4 & -4 \le x \le 0 \\ x-4 & 0 \le x \le 4, \end{cases}$ and

$$\varphi_2(x) = \begin{cases} x+4 & -4 \le x \le 0 \\ -x+4 & 0 \le x \le 4. \end{cases}$$

Note that on the interval $[-4,4]$

$\varphi_2(x) \ge \varphi_1(x)$.

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Then $\iint_D f(x,y)\,dA = \int_{-4}^{4}\int_{\varphi_1(x)}^{\varphi_2(x)} x^2\,dy\,dx$

$$= \int_{-4}^{0}\int_{-x-4}^{x+4} x^2\,dy\,dx + \int_{0}^{4}\int_{x-4}^{-x+4} x^2\,dy\,dx$$

$$= \int_{-4}^{0}\left(x^2 y\Big|_{-x-4}^{x+4}\right)dx + \int_{0}^{4}\left(x^2 y\Big|_{x-4}^{-x+4}\right)dx$$

$$= \int_{-4}^{0}\left(x^2(x+4) - x^2(-x-4)\right)dx$$
$$+ \int_{0}^{4}\left(x^2(-x+4) - x^2(x-4)\right)dx$$

$$= \int_{-4}^{0}(x^3+4x^2+x^3+4x^2)\,dx + \int_{0}^{4}(-x^3+4x^2-x^3+4x^2)\,dx$$

$$= \int_{-4}^{0}(2x^3+8x^2)\,dx + \int_{0}^{4}(-2x^3+8x^2)\,dx$$

$$= \left(\frac{x^4}{2} + \frac{8}{3}x^3\right)\Big|_{-4}^{0} + \left(-\frac{x^4}{2} + \frac{8}{3}x^3\right)\Big|_{0}^{4}$$

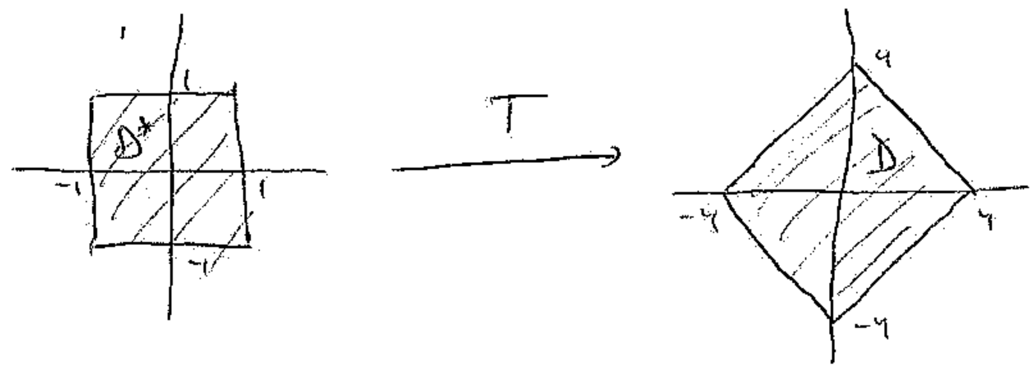
$$= -128 + \frac{512}{3} + -128 + \frac{512}{3} = -256 + \frac{1024}{3}$$
$$= \frac{256}{3}$$

II Consider the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(u,v) = (2u+2v,\ 2u-2v).$$ Here

$$(1,1) \xmapsto{T} (4,0), \quad (1,-1) \xmapsto{T} (0,4)$$
$$(-1,1) \longmapsto (0,-4), \quad (-1,-1) \xmapsto{T} (-4,0)$$

Then the transformation $T: D^* \to D$



By the Change of Variables Formula, we get

$$\iint_D f(x,y)\,dA = \iint_{D^*} f(x(u,v),\, y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} (2u+2v)^2 \left| \begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix} \right| du\,dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} 8\,(4u^2+8uv+4v^2)\,du\,dv$$

$$= 8\int_{-1}^{1} \left(\tfrac{4}{3}u^3 + 4u^2v + 4uv^2\right)\Big|_{-1}^{1} dv = 8\int_{-1}^{1}\left(\tfrac{4}{3} + 4v + 4v^2 + \tfrac{4}{3} - 4v + 4v^2\right) dv$$

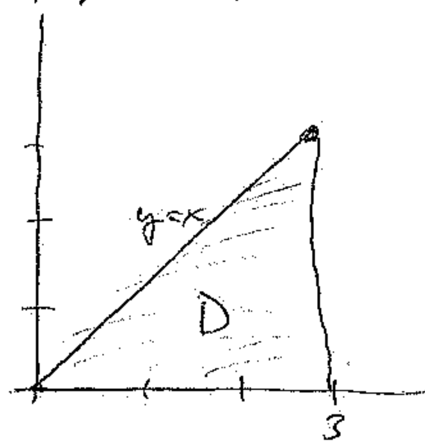
$$= 8\int_{-1}^{1}\left(\tfrac{8}{3} + 8v^2\right) dv = 8\left(\tfrac{8}{3}v + \tfrac{8}{3}v^3\right)\Big|_{-1}^{1} = 8\left(\tfrac{8}{3}+\tfrac{8}{3}+\tfrac{8}{3}+\tfrac{8}{3}\right) = \tfrac{256}{3}$$

Note: ① Often we change coordinates to "fix" the domain. But sometimes we change coordinates to simplify the <u>integrand</u>:

ex. Evaluate $\displaystyle\int_0^3\int_0^x \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$.

<u>Strategy</u>: We could use a trig substitution to do the nested inside integral. Instead we notice $r=\sqrt{x^2+y^2}$ and switch to polar to simplify the integrand.

<u>Solution</u>:



Let $T:\mathbb{R}^2\to\mathbb{R}^2$,

$$(r,\theta)\xrightarrow{T}(\underbrace{r\cos\theta}_{x},\underbrace{r\sin\theta}_{y})$$

Then by the Thm above.

$$\int_0^3\int_0^x \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\theta}} \frac{1}{r}\,r\,dr\,d\theta$$

This is true since ① the region D in polar coordinates ranges from 0 to $\frac{\pi}{4}$ in θ, ex. Solution (cont'd.)

and from $r=0$ to the vertical line whose equation in polar is $r\cos\theta = 3$. Hence r ranges from $r=0$ to $r = \frac{3}{\cos\theta} = 3\sec\theta$.

Hence
$$\int_0^3\int_0^x \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^{\frac{\pi}{4}}\int_0^{3\sec\theta} dr\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \left(r\Big|_0^{3\sec\theta}\right) d\theta$$
$$= \int_0^{\frac{\pi}{4}} 3\sec\theta\,d\theta$$
$$= 3\ln|\sec\theta + \tan\theta|\Big|_0^{\frac{\pi}{4}}$$
$$= 3\ln\left|\tfrac{2}{\sqrt{2}}+1\right| - 3\ln|1+0|$$
$$= 3\ln|\sqrt{2}+1|. \quad \blacksquare$$

You should solve this integral the original way to check.